Renormalization Group Approach to Chaos

(Creswick Chapter 2)
(Strogatz (3rd Ed) Chapter 10 Nonlinear Dynamics and Chaos)

We will now discuss one of the most stunning and novel examples of RG... In the late 1970s, Mitch Fiegenbaum developed an "Functional RG" approach to understand the period doubling approach to chaos...

Let us start by first describing the phenomenon...

Consider a 1-d discrete dynamical map

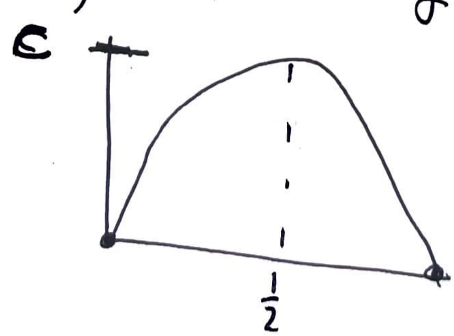
$$x_{n+1} = f(x_n, c)$$

Paramater

$$f(x,c) = 4c\,x(1-x) \quad [\text{Logistic}]$$

$$f(x,c) = c \sin \pi x$$

More generally consider any $f(x)$ that has maximum at $x = 1/2$ and is zero at $x=0$ and $x=1$ (can map any single peaked function to this by rescaling... x-axis..)



We will be interested in characterizing the qualitative changes in dynamics...

At long times this is a question about fixed points

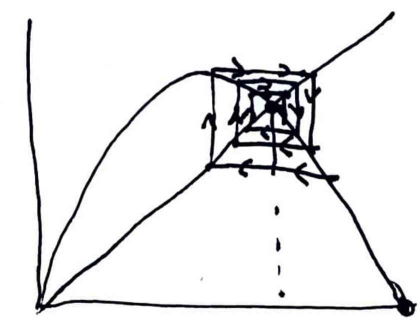
$$x_0^* = f(x_0^*, c)$$

$$x_0^* = 4c\,x_0^*(1-x_0^*)$$

$$\left\{ \frac{1}{4c} = 1 - x_0^* \right.$$

$$x_0^* = 1 - \frac{1}{4c}$$

Graphically



x_0

For c sufficiently small this is ~~a fixed~~ fixed point is attractor of dynamics (as shown above) as long as small perturbations $x_0^* + \delta x$ decay...

We see this just translates into the statement

$$|f'(x_0^*, c)| < 1$$

So at some c_0 the map becomes unstable

$\Rightarrow$ For logistic

$$f'(x_0^*, c) = 4c(1-2x)$$

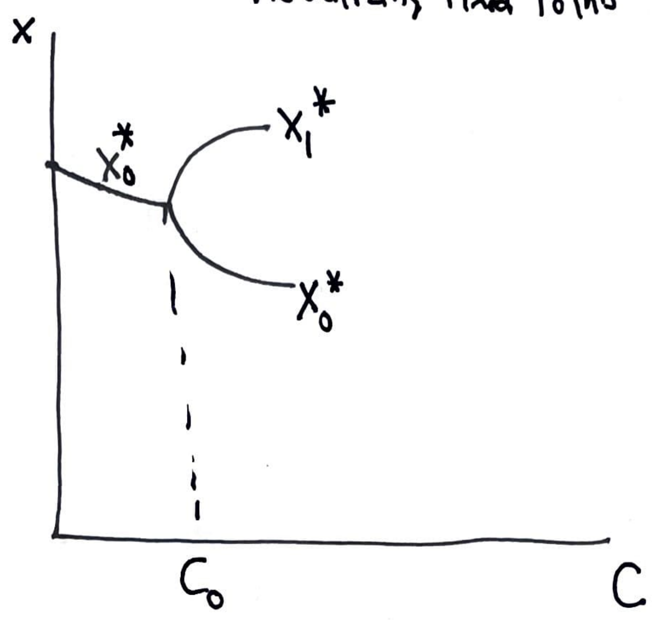
$$= 4c\left(-1 + \frac{1}{2c}\right)$$

$$-1 = -4c + 2$$

Notice that for some $c_0 = \frac{3}{4}$

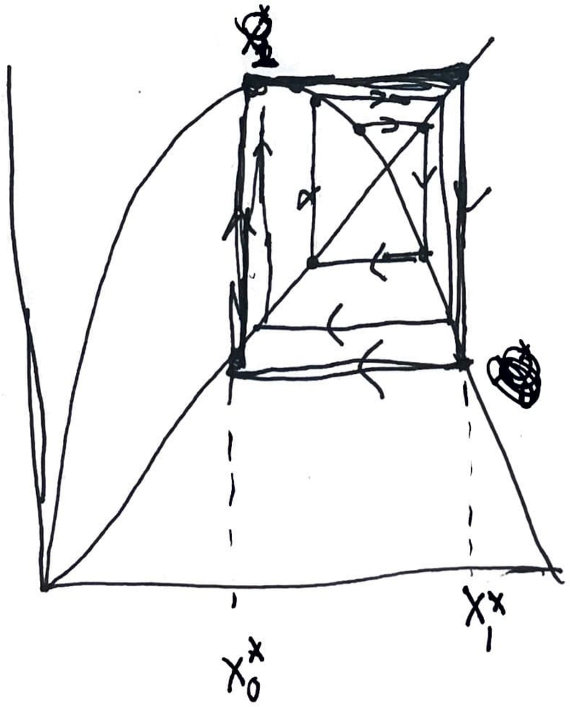
What happens next is really intersting, there is no ~~period 2~~ fixed point but a bifurcation to a "period 2" attractor

Visualizing Fixed Points



where

$$x_1^* = f(x_0^*, C)$$
$$x_0^* = f(x_1^*, C)$$



Amazingly notice that we can make a new map

$$f^{(2)}(x,C) = f(f(x,C))$$

We see that x_0^* and x_1^* are both fixed points of $f^{(2)}(x,C)$.

By the same argument these will go unstable when

$$\left| \frac{\partial f^{(2)}}{\partial x}(x,c) \right|_{x = x_0^*, x_1^*} > 1$$

This happens when

$$\left.\frac{\partial f^{(2)}(x,c)}{\partial x}\right|_{x_0} = \frac{\partial f}{\partial x}(f(x_0^*,c))\frac{\partial f}{\partial x}(x_0^*,c) = \frac{\partial f}{\partial x}(x_1^*,c)\,\frac{\partial f}{\partial x}(x_0^*,c)$$

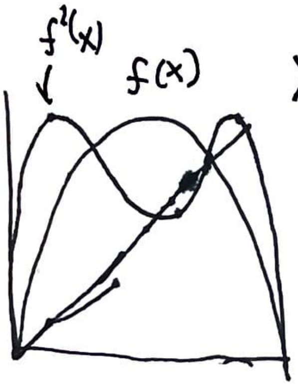
Notice $f^{(2)}(x,c)$ has two stable ~~period~~ fixed points x_0^*, x_1^* (period 1 orbits)

So now each of these fixed points will split at some C_1

More generally it is clear that if I have same function by composition

$$f^4(x,c) = f^{(2)}(f^{(2)}(x,c)) \rightarrow$$

We can run this argument over and over again



$$x_0^* = f(x_1^*,c)$$
$$x_2^* = f(x_1^*,c)$$
$$x_3^* = f(x_2^*,c)$$
$$x_0^* = f(x_3^*,c)$$

More generally we can consider

$f^{(2^n)}(x,c)$ which will have ~~stable~~

2^n fixed point (also called period 1 orbits)

By construction those fixed points are the period 2^n – orbit of $f(x)$

Furthermore we see that by almost analogous argument we made for $f^2(x)$ that

$$\frac{d}{dx} f^{(2^n)}(x,c)\Big|_{x=r^*} = \underbrace{\prod_{k=0}^{2^n-1} f'(x_k^*, c)}$$

Product runs over all fixed points of $f^{(2^n)}$ or equivalently all members of 2^n-period orbit of $f(x_k, c)$

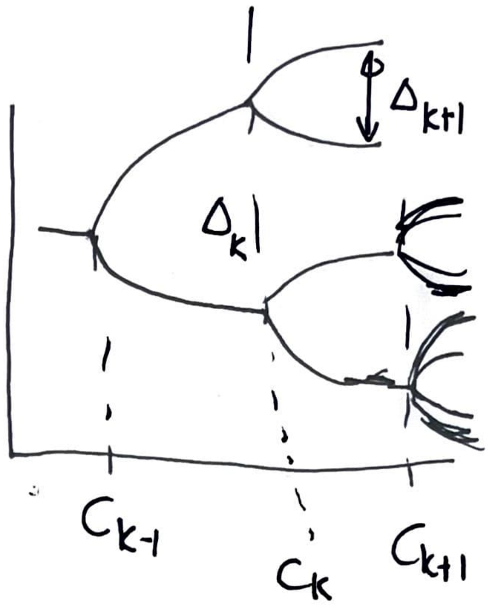
What is cool is that the distance between bifurcations decreases as power law
(in paramater space)

$$\lim_{k\to\infty} c_\infty - c_k \sim \delta^{-k}$$

(Appearance of chaos) ≈ 3.57 for Logistic

and spacing between k and $k+1$ st braches of tree tend to finite limit

$$\lim_{k\to\infty} \frac{\Delta_k}{\Delta_{k+1}} \to \alpha$$



Furthermore these exponents are <u>universal</u>
do not depend on logistic map

$$\alpha = 2.50290787$$

$$\delta = 4.6692016\ldots$$

(Feigenbaum constants)

This all looks self similar so we should able to use RG to calculate this...

And well will, we use a variant of RG we call Functional RG.

The idea of functional RG is to speak about RG transformations in space of functions... Not so different from thinking about parameters

$$\{K_j\} \longrightarrow \{K_j'\}$$

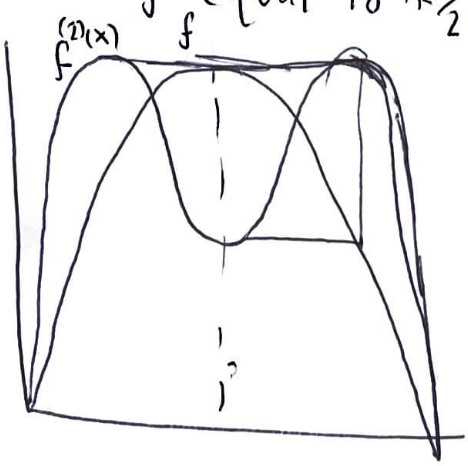
Since equivalent to transforming Hamiltonian functions

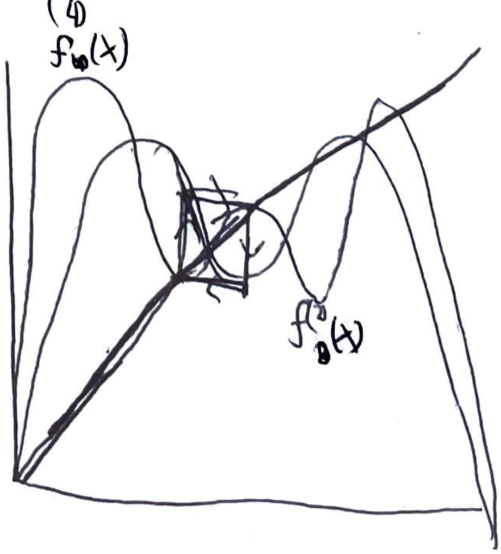
$$H(\{K_j\}) \longrightarrow H(\{K_j'\})$$

Recall that RG is two steps

1. Coarse Graining
2. Scale transformation

Instead of focusing of place where bifurcation we will focus on point at which one of fixed points becomes exactly equal to $x=\frac{1}{2}$





Call this poin λ_n for $f^{(2^n)}(\frac{1}{2}, \lambda_n) = \frac{1}{2}$

What is interesting is that the two figures in the vicinity of $X = \frac{1}{2}$ that the two "circulating square" look almost the same except reflected and reduce in size by factor α_n. If we define

$$d_n = f^{(2^{n-1})}\left(\frac{1}{2}, \lambda_n\right) - \frac{1}{2}$$

Then we have

$$\alpha_n = \frac{d_n}{d_{n+1}} = \frac{f^{(2^{n-1})}\left(\frac{1}{2}, \lambda_n\right) - \frac{1}{2}}{f^{(2^n)}\left(\frac{1}{2}, \lambda_n\right) - \frac{1}{2}}$$

(minus sign is reflection from left to right)

Notice we have coarse srained by just focusing of this region

This is approximate self-similarity, so we would like to rescale

So let us define $y = X - \frac{1}{2}$ and function..

$$g_r(y) = \lim_{n \to \infty} (-\alpha)^n f^{(2^n)}\left(\frac{y}{\alpha^n}, \lambda_{n+r}\right)$$

"You will show for HW with this definition

$$g_{r-1}(y) = -\alpha\, g_r\left(g_r\left(\frac{x}{\alpha}\right)\right)$$

In particular we can define

$$g(y) = \lim_{r\to\infty} g_{r-1}(y)$$

We know that

$$g(y) = -\alpha\, g\left(g\left(\frac{y}{\alpha}\right)\right).$$

Notice in addition (HW problem) that if $g(y)$ is solution to this equation so is $g_\mu(y) = \mu\, g\left(\frac{y}{\mu}\right)$.

This allows us to rescale g so that WLOG we assume (fix gauge) $g(0) = 1$.

Then we have

$$g(0) = -\alpha\, g(g(0))$$
$$1 = -\alpha\, g(1)$$

$$g(1) = -\frac{1}{\alpha}$$

This allows us to define a RG transformation

$$T[\psi](x) = -\alpha\, \psi\left(\psi\left(\frac{x}{\alpha}\right)\right)$$

Maps function to anothe fruction..

Notice

$$g_{r-1}(x) = T[g_r](x)$$

So we see that we can repeatedly apply so that

$$\infty \to g_r(x) \to g_{r-1}(x) \to \ldots \to g_2(x) \to g_1(x)$$

This shows fixed point $g(x)$ is unstable and this is relevant perturbation

Let us calculate the scaling exponent (e.v.) of linearized map

$$L[\psi](x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left\{T[g+\epsilon\psi](x) - g(x)\right\}$$

$$L[\psi](x) = -\alpha\left[g'\left(g\left(\frac{x}{\alpha}\right)\right)\psi\left(\frac{x}{\alpha}\right) + \psi\left(g\left(\frac{x}{\alpha}\right)\right)\right]$$

To see this notice

$$T[g+\epsilon\psi](x) - g(x) = \left[-\alpha\, g\left(g\left(\tfrac{x}{\alpha}\right) + \epsilon\psi\left(\tfrac{x}{\alpha}\right)\right) - \alpha\epsilon\psi\left(g\left(\tfrac{x}{\alpha}\right) + \epsilon\psi\left(\tfrac{x}{\alpha}\right)\right)\right] - g(x)$$

Expand to linear in ε

$$\underbrace{-\alpha g\left(g\left(\tfrac{x}{\alpha}\right)\right)}_{g(x)} - \epsilon\alpha g'\left(g\left(\tfrac{x}{\alpha}\right)\right)\psi\left(\tfrac{x}{\alpha}\right) - \alpha\epsilon\psi\left(g\left(\tfrac{x}{\alpha}\right)\right) - g(x)$$

So linear term

$$L[\psi](x) = \alpha g'\left(g\left(\tfrac{x}{\alpha}\right)\right)\psi\left(\tfrac{x}{\alpha}\right) - \alpha\psi\left(g\left(\tfrac{x}{\alpha}\right)\right)$$

Define

$$g_r(x) = g(x) - \Delta g_r$$
$$g_{r-1}(x) = g(x) - \Delta g_{r-1}$$

$$g_{r-1}(x) = T[g_r(x)] = T[g(x) - \Delta g_r] = -L[\Delta g_r] + g(x)$$

$$g(x) - \Delta g_{r-1} = g(x) - L[\Delta g_r]$$

$$\Delta g_{r-1} = L[\Delta g_r]$$

So now ~~l~~et us assume that

$$\Delta g_r(x) = p_r h(x)$$

["Sep of variables" or scaling ansatz]

$$p_r L[h](x) = p_{r-1} h(x)$$

So that since it has to hold for all ~~x~~ that

$$L[h](x) = \delta h(x)$$

where

$$p_r \sim \delta^{-r}$$

So let us idetinfy this with Feigenbaum number

$$\lim_{k \to \infty} \delta_k \approx \left. \frac{c_k - c_{k-1}}{c_{k+1} - c_k} \right\}$$

ratio of successive bifurcations..

First notice

$$g_r(x) = g(x) - \delta^{-r} h(x)$$

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To proceed we note that we need to show

$$\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} \sim \delta \quad \left(\begin{array}{l} \text{A little different than} \\ \dfrac{C_n - C_{n-1}}{C_{n+1} - C_n} \text{ but since} \\ \text{bounded same thing} \end{array} \right)$$

To do this we will use two facts. Recalling that

$$f^{(1)}(x=\tfrac{1}{2}, C) = C \quad \text{and in particular} \quad f^{(1)}(y=0, \lambda_n) = \lambda^n$$

and

$$g_r(y) = \lim_{n\to\infty} (-\alpha)^n f^{(2^n)}\left(\frac{y}{\alpha^n}, \lambda_{n+r}\right)$$

We have that we can make since $g_r(x) = g(x) - \delta^{-r} h(x)$

$$g_{r+1}(y) - g_r(y) = +\delta\,[g_r(y) - g_{r-1}(y)]$$

$$= \delta^{-1}\left[\lim_{n'\to\infty} (-\alpha)^{n'}\left[f^{(2^{n'})}\left(\frac{y}{\alpha^{n'}}, \lambda_{n'+r}\right) - f^{(2^{n'})}\left(\frac{y}{\alpha^{n'}}, \lambda_{n'+r-1}\right)\right]\right]$$

Let us write $n' = n+1$

$$\lim_{n\to\infty} f^{(2^n)}\left(\frac{y}{\alpha^n}, \lambda_{n+r+1}\right)\alpha^n - f^{(2^n)}\left(\frac{y}{\alpha_n}, y_r\right) = \frac{\alpha}{\delta} \lim_{n\to\infty} \alpha^n \left[f^{(2^{n+1})}\left(\frac{y}{\alpha^n}, \lambda_{n+r+1}\right) - f^{(2^{n+1})}\left(\frac{y}{\alpha^n}, \lambda_{n+r}\right)\right]$$

For $y=0$ we have recusion relation

<u>In n</u>

$$f^{(2^{n+1})}(0, \lambda_{n+r+1}) - f^{(2^{n+1})}(0, \lambda_{n+r}) = \left(\frac{\delta}{\alpha}\right)\left[f^{(2^n)}(0, \lambda_{n+r+1}) - f^{(2^n)}(0, \lambda_{n}\ldots\right.$$

This gives us that

$$f^{(2^n)}(0, \lambda_{n+r+1}) - f^{(2^n)}(0, \lambda_{n+r})$$
$$= \left(\frac{\delta}{\alpha}\right)^n \left[f^{(1)}(0, \lambda_{n+r+1}) - f^{(1)}(0, \lambda_{n+r}) \right]$$

So that
$$f^{(2^n)}(0, \lambda_{n+r+1}) - f^{(2^n)}(0, \lambda_{n+r}) \simeq \left(\frac{\delta}{\alpha}\right)^n \left[\lambda_{n+r+1} - \lambda_{n+r}\right]$$

Multiplying by α^n

$$g_{r+1}(0) - g_r(0) = \delta^n \left[\lambda_{n+r+1} - \lambda_{n+r}\right]$$

Which implies for n large

$$\frac{\lambda_{n} - \lambda_{n+1}}{\lambda_{n-1} - \lambda_n} = \delta$$

This is what we wanted...