Lecture 3: Universality of RW / S.A.W RG

(Cresswick 1.7, 1.8)

Thus far we have considered the special case where the RW lengths were drawn from a Gaussian

$$p(\vec{r}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|r|^2}{2\sigma^2}}$$

However, one of the most amazing things about Stat Mech and RG is the idea of universality... Many systems with different short distance physics have long distance physics that is the same...

So let us look at the more general class of RWs where

$$p(\vec{r}) = p(|r|)$$ (i.e. depends only on magnitude not direction)

This includes Gaussian and $p(\vec{r}) = \frac{1}{4\pi a^2}\delta(|r| - a)$
fixed size

Let us again coarse grain

$$\vec{r}' = \sum_{i=1}^{n} \vec{r}_i$$

As before

$$P(\vec{r}') = \int d\vec{r}_1^{\,d} \ldots \int d^d\vec{r}_n \, \delta(\vec{r}' - \sum_{i=1}^{n} \vec{r}_i) \, p(\vec{r}_1) \ldots p(\vec{r}_n)$$

To proceed we will introduce a Fourier transform

$$g(\vec{k}) = \int d^d r \, e^{i\vec{k}\cdot\vec{r}} p(r) = \langle e^{i\vec{k}\cdot\vec{r}} \rangle$$

Notice that this is just almost partition function on $\beta \to i\vec{k}$

$\langle e^{\lambda A} \rangle \backsim$ is like partition fuction } Generating function

$\log \langle e^{\lambda A} \rangle \backsim$ like Free Energy } Cumulant Generating Function...

$$\ln \langle e^{i\vec{k}\cdot\vec{r}} \rangle \approx ik\langle \vec{r} \rangle - \frac{1}{2}\sum k_\mu k_\nu \overbrace{\underbrace{(\langle r_\mu r_\nu \rangle - \langle r_\mu \rangle \langle r_\nu \rangle)}_{\text{2nd cummulant / connecte correlation function}}}^{\sigma^2_{\mu\nu}}$$

$$+ \ldots$$

By symmetry $\langle \vec{r} \rangle = 0$ and

$$\sigma^2_{\mu\nu} = \langle r_\mu r_\nu \rangle - \langle r_\mu r_\nu \rangle = \sigma_0^2 \delta_{\mu\nu}$$

$$p(\vec{r}') = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\cdot\vec{r}'} \prod_{j=1}^{n} \int dr_j^d \, e^{-i\vec{k}\cdot r_j} p(\vec{r}_j)$$

$$= \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\cdot\vec{r}'} \prod_{j=1}^{n} g(\vec{k})$$

$$\backsim \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\cdot\vec{r}'} e^{-\frac{n}{2}\sigma^2 k^2 + \underbrace{\ldots\ldots}_{\text{higher order cumulants}}}$$

Now we will assume $n \gg 1$ ... So we can use "saddle point" or.. (neglect higher order cumulants)

Since mth cumulant scales as

$$k^m \sim n^{-m/2}$$

$k \sim \frac{1}{\sqrt{n}\sigma_0^2}$ contribute.

So contribution goes like

$$n \cdot k^m \sim n^{1-m/2}$$

Thus the coarse grain distribution is just

$$p(\vec{r}') = \frac{1}{(2\pi n \sigma_0)^{d/2}} e^{-\frac{1}{2} \frac{|\vec{r}'|^2}{n\sigma_0^2}}$$

This is same thing we got for Gaussian so Same scaling / RG argument goes through

... So all these RW belong to same universality class ... that of Gaussians where

$$R \sim N^\nu \qquad \nu = \tfrac{1}{2}$$

One of the most interesting examples of a Random Walk that belongs to a different universality class is the self-avoiding random walk

$$R = N^{\nu}$$

$\nu = 0.74 - 0.75$ (conjectured to be $3/4 = \frac{3}{d+2}$) for SLE)

proven in $d \geq 5$

exact in $d=2$

thought to be

$d=4$ logarithmic correction

$d=3$ 0.58...

Self avoidance is a complicated geometrically so we cannot do anything in closed form..

Start by defining

$M_n(\vec{R})$ be the number of SAWs on n steps that ~~connect~~ end at $\vec{R}$.

The mean squared displacement for walks of lenth n is just

$$\langle R_n^2 \rangle = \frac{\sum_R |\vec{R}|^2 M_n(\vec{R})}{\sum_n M_n(\vec{R})}$$

Actually instead of working with fixed stepsize n,
... Insted work in the grand cannonical ensemble ...
Recall transformation ..

$$Q(z) = \sum_n z^n Z_n = \sum_n e^{\mu n} Z_n$$

Analogy define generating function ...

$$G(k) = \sum_n k^n \sum_R M_n(\vec{R}) \quad \Big(k - \text{fugacity} \ldots \rightarrow e^{\mu} \text{ like chemical potential}\Big)$$

We can define correlation length

$$\xi^2(k) = \frac{\sum_R \sum_n |R|^2 M_n(R) k^n}{G(k)}$$

This is the length that measures typical size of a walk for given fugcity K...

$M_n(R)$ rapidly increasing function of n
for $k<1$ k^n is rapidly decreasing function of n.

So we expect that there will be a sharp maximum for particular value

$$n = n^*(k)$$

as k increases so does $n_o(k)$ and finally as K approaches a critical K_c, $n^*(K)$ diverges and $G(K)$ becomes singular...

$$K \leq K_c$$ (this is where we can learn about infinite SAW)

We assume that for large n, we assume scaling...

$$M_n \sim K_c^{-n}\, n^{\gamma-1}$$ (see below)

$$K^n M_n = \exp\left(\underbrace{n \ln\frac{K}{K_c} + (\gamma-1)\ln n}\right)$$

This maximal (peaked) at

$$\ln\frac{K}{K_c} + \frac{(\gamma-1)}{n^*} = 0$$

$$n^*(K) = \frac{\gamma-1}{\ln\left(\frac{K_c}{K}\right)}$$

as $K \to K_c$, $n^*(K)$ become

$$K = K_c - \delta K \qquad \ln\left(\frac{K_c}{K_c - \delta K}\right) = -\ln\left(1 - \frac{\delta K}{K_c}\right)$$

$$\simeq \frac{\delta K}{K_c}$$

$$n^*(K) \to (\gamma-1)\left(\frac{K_c}{K_c - K}\right)$$

So then we see that in also assume

then $R^2 \sim n^{\nu}$

$$\xi^2(K) \sim n_0^{\nu}(K) \sim \left(\frac{K_c - K}{K_c}\right)^{-\nu}$$

So we see that the tyical length of SAW as $K \to K_c$ diverges with critical exponent ν

We will formulate RG for $G(K)$

$\boxed{\text{RG in Grand Cannonical}}$

So far we introduced a generating function $G(K)$ that just counts number of steps or "bonds"

But we can introduce weight K_2 for each pair on next-neighbor sites, on weigh K_3 for third neighbor, ...

So then statistics of SAW characterized by weights $\{K_i\}$ and a generalized weight function

$W(\Gamma, \{K_i\})$ associated with particular realization Γ of a SAW

So the RG transformation takes

$$\Gamma \xrightarrow[\text{lattice}]{\text{coarse grained}} \Gamma'$$

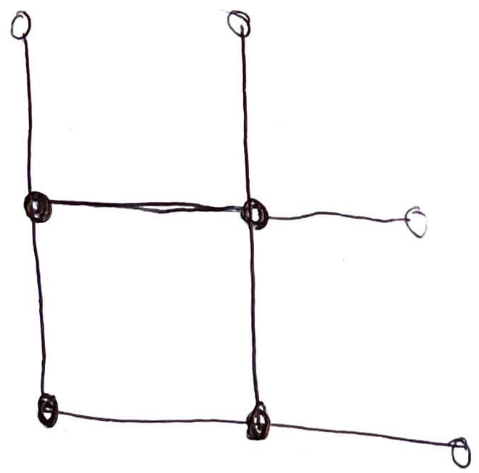
(in general many walks map to same Γ')

We now require statistical weights associated with Γ' must equal the statistical weight of the walks that "map" to Γ'

$$W(\Gamma', \{K_i'\}) = \sum_{\Gamma \in \Gamma'} W(\Gamma, \{K_i\})$$

So what we will do is coarse grain

2 step $\to$ 1 step



$b = 2$

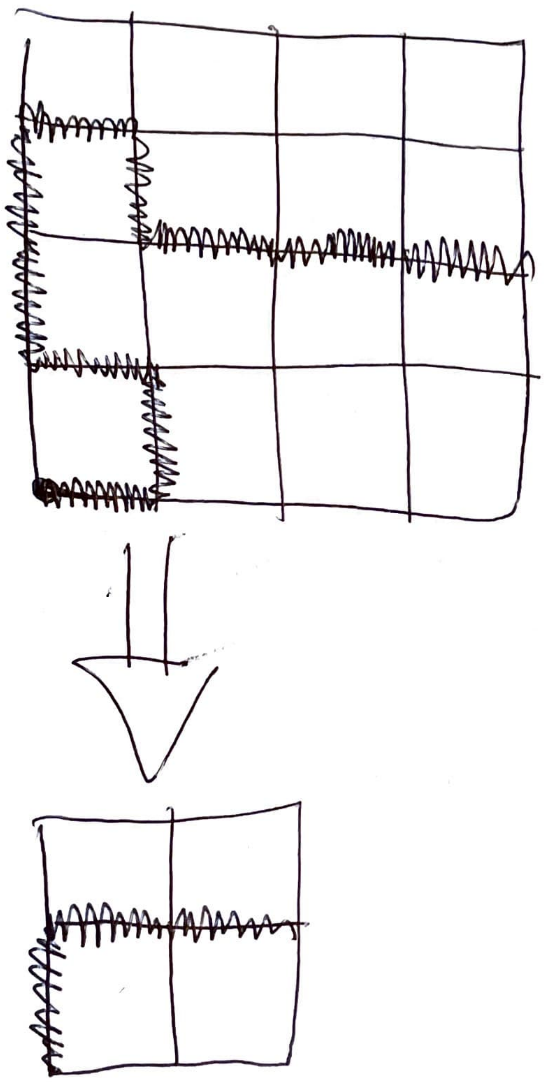
So something with fugacity K in region b mapped to a SAW with fugacity K' on coarse grained lattice

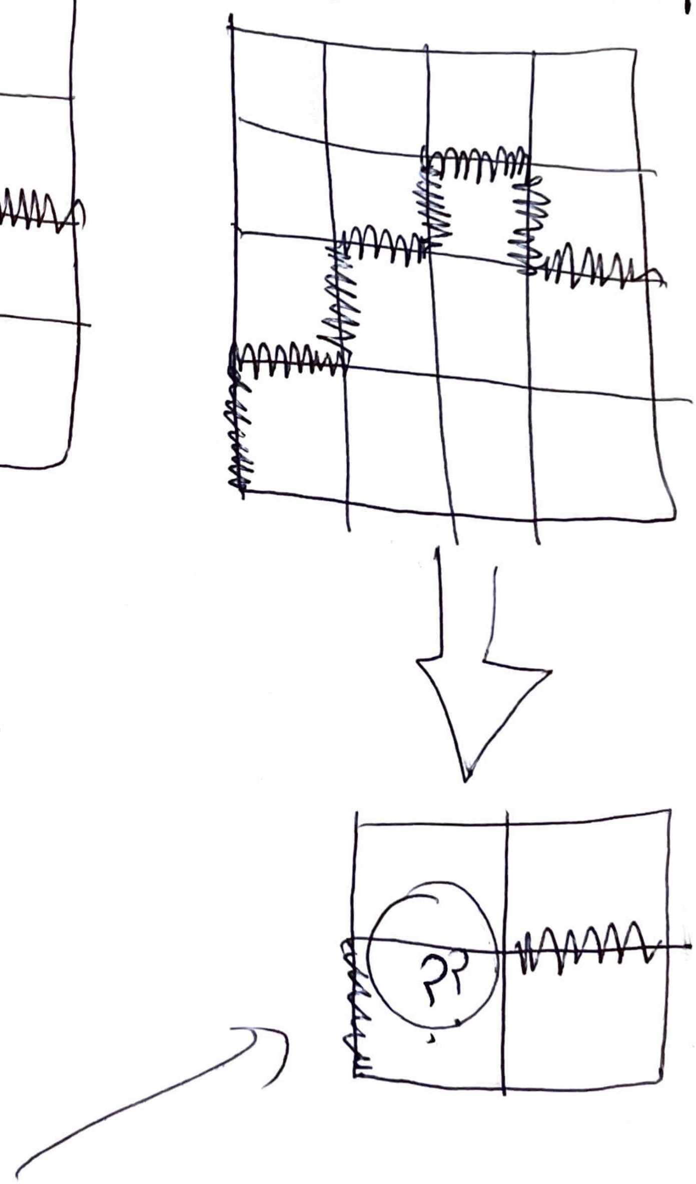
$K \to K'$

~~But as we will see this~~

$$\xi(k) = b\,\xi'(k')$$

Let us construct an RG transformation we call corner rule... (exits through top of 2x2 block its "up step" or righ its "right step"

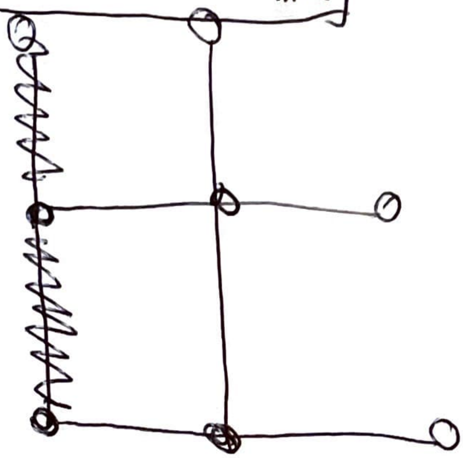




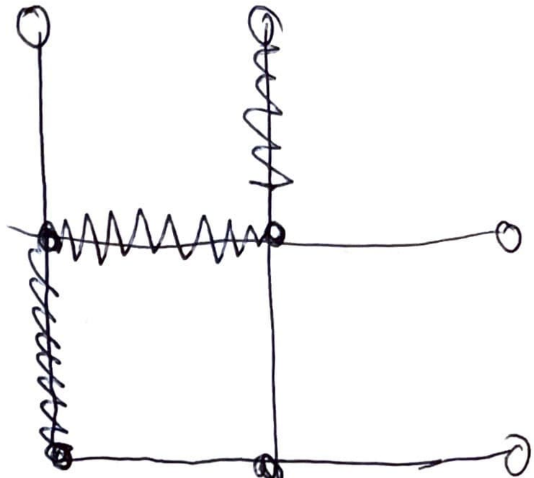
This is ambiguous...

We actually have to keep track of K_2 and K_3 ... the kinds of two step and 3 step walks

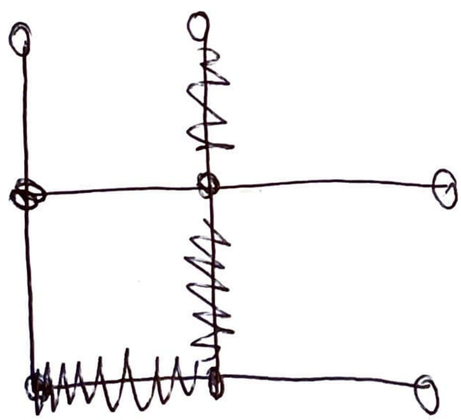
Vertical Step Walks



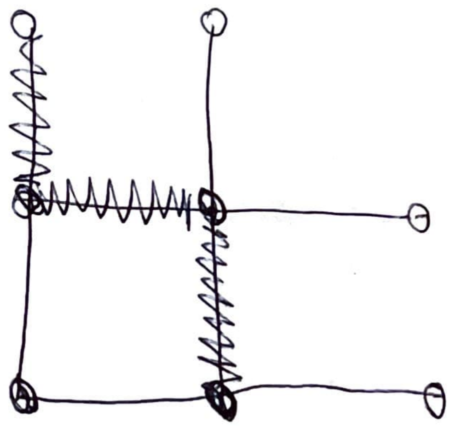
K^2



K^3



K^3

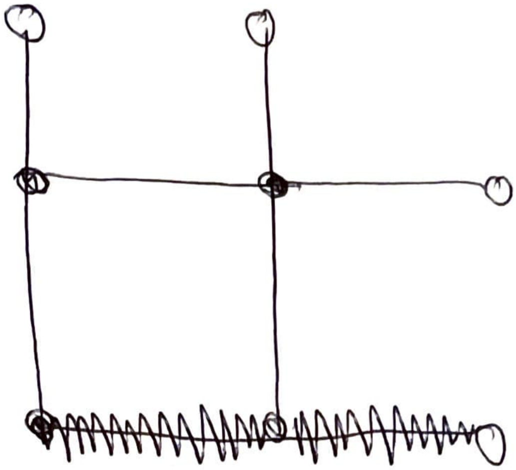


K^4

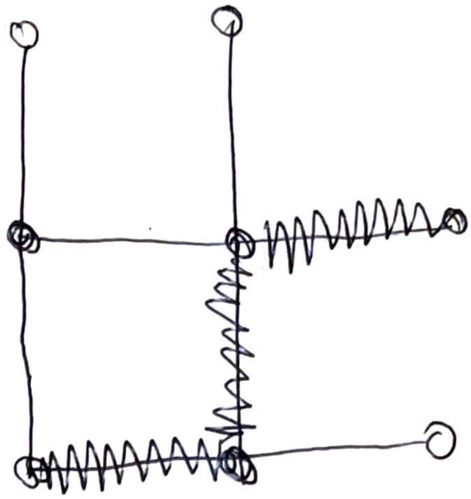
$$K' = K^4 + 2K^3 + K^2$$

(Obviously for horizontal we have

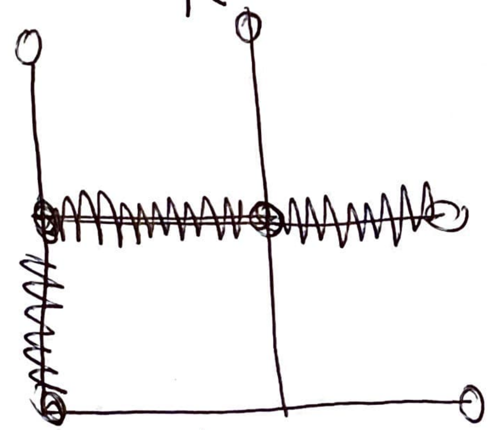
Horizontal



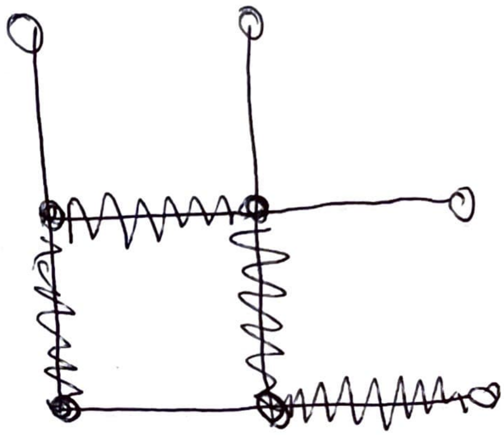
K^2



K^3



K^3



K^4

$$K' = K^2 + 2K^3 + K^4$$ (Same equation so consistent...)

The fixed points of this equation are

$K^* = 0$

$K^* = \infty$

and $K^* = K_c = 0.4655...$

So now to get critical exponent...

Assume

$$\xi(K) \sim |K - K_c|^{-\nu}$$

and recall by dimensional analysis

$$\xi(K) = b\,\xi(K')$$

Under linearized RG...

$$K' - K_c = \left|\frac{\partial K'}{\partial K}\right|_{K=K_c} (K - K_c)$$

$$|K' - K_c|^{-\nu} = \left[\frac{\partial K'}{\partial K}\bigg|_{K_c}\right]^{-\nu} |K - K_c|^{-\nu}$$

$$\xi(K') = \left[\frac{\partial K'}{\partial K}\bigg|_{K_c}\right]^{-\nu} \xi(K)$$

$$\frac{1}{b} = \left[\frac{\partial K'}{\partial K}\bigg|_{K_c}\right]^{-\nu}$$

$$\nu = \frac{\ln b}{\ln \left|\frac{\partial K'}{\partial K}\right|_{K_c}}$$

$K_c = 0.4655$

$\nu = 0.715$

Compare exact

$K_c = 0.379$ $\quad \nu = 0.74 - 0.75$