Lecture : Self-Similarity and Scale Invariance ⇒ RW.
(Random Walks)

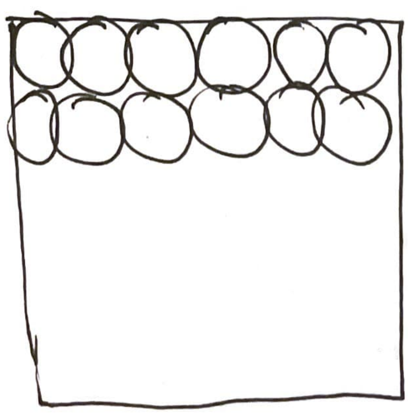
As we discussed last time, a central role is played fixed points which are invariant under scale transformation.... In other words, they are self-similar.... (scale invariance at all scales)

Such self-similar structures are often "fractal" and defined by a fractal dimension

To understand this

D – fractal or Hausdorff dimension

Consider an object (say the plane in $\mathbb{R}^2$)



Let $N(a)$ be the minimum number of d-dim. balls needed to cover the object (min is over positions

$$N(a) \sim a^{-D}$$

(agrees with usual idea of dimension for objects

Notice that

$$d_T \leq D \leq d$$

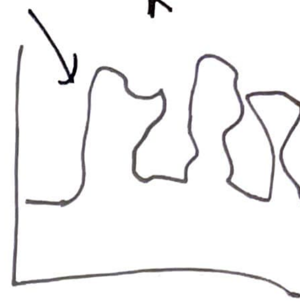
Topological dimension

(1 for curve)

$(\mathbb{R}^2)$

dimension of embedding space

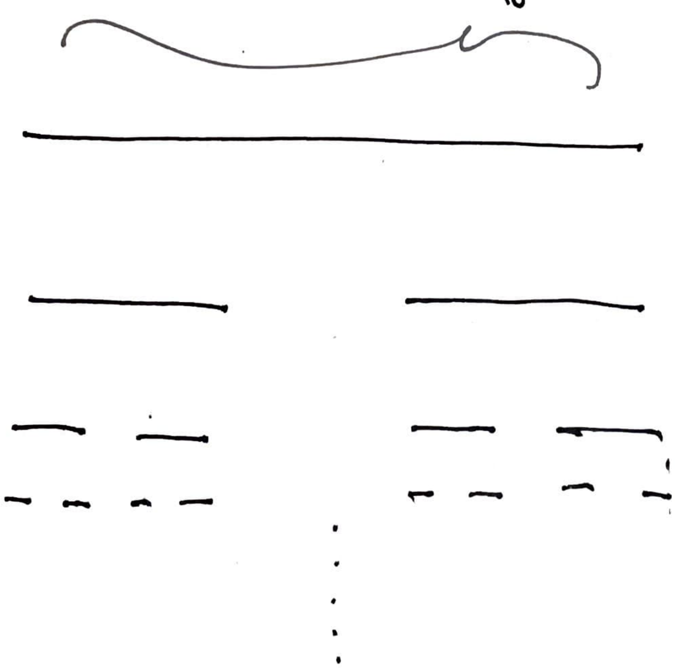
Curve in $\mathbb{R}^2$



So how can we create objects where $D \neq d_T, d$

Classic example is "Cantor Set"

a_0



Take a line segment $d=1$ of length a_0

Remove middle third

Repeat

Now choose $a = a_0 3^{-n}, \quad n = 1, 2, \ldots$

$$N(a) = 2^n = 2^{-\log_3 \frac{a}{a_0}} = 2^{-\frac{\ln(a/a_0)}{\ln 3}} = e^{-\left(\frac{\ln 2}{\ln 3}\right)\ln\left(\frac{a}{a_0}\right)}$$

$$= \left(\frac{a}{a_0}\right)^{-\frac{\ln 2}{\ln 3}} \Rightarrow D = \frac{\ln 2}{\ln 3} = 0.6309\ldots$$

As promised $d_T = 0$ (points)

$d = 1$ (embedded in 1 D)

$$D = 0{,}6309 = \frac{\ln 2}{\ln 3}$$

Useful to reformulate this using self-similarity recusively

$$N(a) = 2N(3a) \longleftarrow$$

Same function

i.e. self similar when we zoom out

Assume scaling

$$N(a) \sim a^{-D}$$

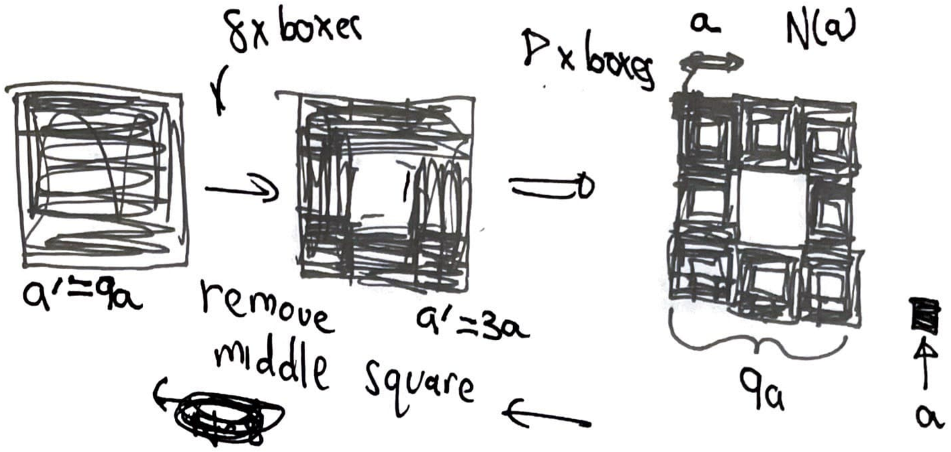
$$a^{-D} = 2(3a)^{-D}$$

$$\Rightarrow 1 = 2 \cdot 3^{-D}$$

$$\Rightarrow \boxed{D = \frac{\ln 2}{\ln 3}}$$

This is a very typical form of an argument we will make...

Consider Sierpinski carpet



$$N(a) = 8\,N(3a)$$

assuming $N(a) \sim a^{-D}$

$$a^{-D} = 8\,(3a)^{-D}$$

$$\log 8 = -D \log 3$$

$$D = \frac{\log 8}{\log 3} = 1.8928\ldots$$

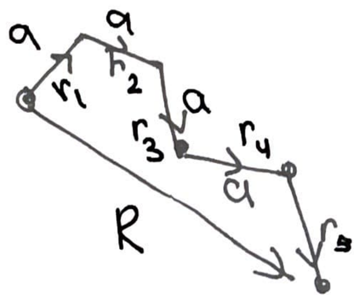
$$d_T \leq D \leq d$$ ← (embedded in 2-dimensions)

We will do this for class....

Perhaps the most famous and simple example of a self-similar object is a random walk (RW)
Show simulation from Mathematica..

Consider a RW in spatial dimension $d \geq 2$

Make M independent step of length r_i or length a



$$\vec{R} = \sum_{i=1}^{M} r_i$$

from independence $\langle r_i \cdot r_j \rangle = 0$ if $i \neq j$

$$\langle |\vec{R}|^2 \rangle = \sum_{i=1}^{M} \sum_{j=1}^{M} \langle r_i \cdot r_j \rangle = \sum_{i=1}^{M} \langle r_i^2 \rangle = M a_0^2$$

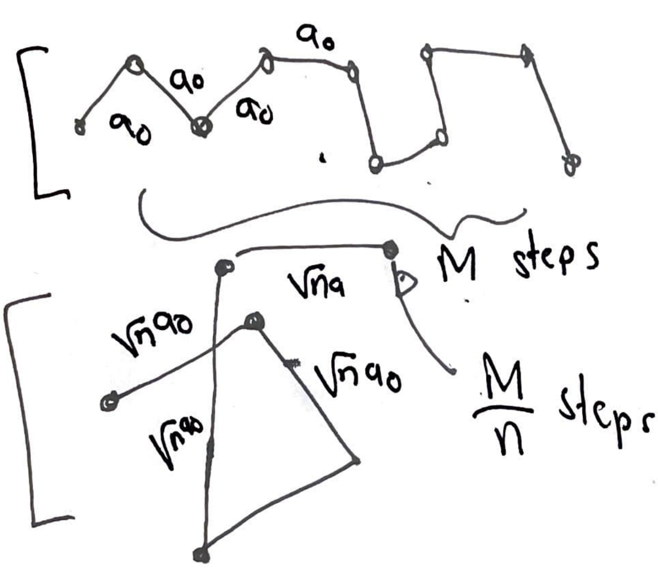
$$R = \sqrt{\langle |\vec{R}|^2 \rangle} \sim \sqrt{M}\, a_0$$

Now assume $M \gg 1$, we can divide random walk into n subwalks of M/n steps...

When $1 \ll n \ll M$, we can make identical argument

$$r(n) = \sqrt{n}\, a_0$$

So notice that R.W. of length M with steps a_0 is self similar to R.W. of length $\frac{M}{n}$ steps each of length $a = \sqrt{n}\, a_0$



Change ruler to

$a = \sqrt{n}\, a_0$

$M \to \frac{M}{n}$

Number of balls of size a_0

So now we have to think about $N(a_0) \sim M$

$$N(\sqrt{n}\, a_0) \sim \frac{M}{n}$$

$$N(a_0) = n\, N(\sqrt{n}\, a_0)$$

for this to be true $\boxed{N(a_0) \sim \frac{1}{a_0^2}}$

fractal dimension

$D = 2$

for any dimension $d \geq 2$!

Revisiting the anatomy of RG Transformation

Before constructing RG

Last time we said RG is a coarse graining procedure but that is not quite right

It actually has two steps

1. Step 1 : Decimation or coarse graining
   $\Rightarrow$ "average out" or "integrate out" or more properly "marginalize" over ~~these~~ short distance degrees of freedom

2. Step 2 : Rescaling $\Rightarrow$ Redefine unit of length $\Rightarrow$ "Use bigger length"
   $\Rightarrow$ Scale factor $b \Rightarrow x' = bx$

   So that smallest distance is restored to original value...

So we have mapped original system back to itself but with short distance physics marginalized over...

This takes the form that the paramaters of the system $\{h_k\}$ $k=1,2,\dots,$

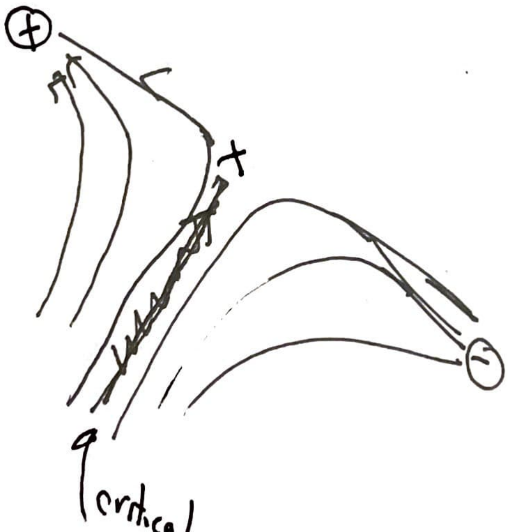
RG equations map to new parameters (important consider space of all possible couplings) to do this properly...

$$h'_k = R_k(\{h_j\})$$

Fixed point: $\lim_{n\to\infty} = R_k(R_k(\ldots n \text{ renormalizations } R_k(h_j^{(c)})) \to h^*_k$

This implies system is self similar...

The special set of of values for which $h_j^{(c)}$ define "critical surface" $\Longrightarrow$ seperatrix in flow space



Now consider

$$h_k = h^*_k + \varepsilon_k \quad \text{with } |\varepsilon_k| \ll 1$$

$$\varepsilon'_k = \sum_j \frac{\partial h'_k}{\partial h_j} \varepsilon_j \qquad \text{Linearize}$$

$$\frac{\partial h'_k}{\partial h_j} = R_{kj} \leftarrow \text{matrix ("Linear operator")}$$

Can of course find right eigenvectors

$$\sum_j R_{ij}\,\phi_j^{(n)} = \lambda^{(n)}\phi_i^{(n)}$$

and left Eigenvectors

$$\sum_j \tilde{\phi}_i^{(n)} R_{ij} = \lambda^{(n)}\tilde{\phi}_i^{(n)}$$

with

$$\sum_k \tilde{\phi}_k^{(n)}\phi_k^{(n')} = \delta_{nn'}$$

and

$$\sum_n \tilde{\phi}_j^{(n)}\phi_k^{(n)} = \delta_{jk}$$

So if we we write

$$\varepsilon = \sum_i U_n \phi_i^{(n)}$$

expand in right eigenvectors

Then under RG

or is marginal $|\lambda^{(n)}| = 0$

$$\varepsilon \to \lambda^{(n)} U_n \phi_i^{(n)}$$

Under repeated transformation
$\phi_i^{(n)}$ coefficient grows $|\lambda^{(n)}| > 1$
shrinks $|\lambda^{(n)}| < 1$

$|\lambda^{(n)}| < 1$ $\phi^{(n)}$ is irrelevant $U_n \to 0$

$|\lambda^{(n)}| > 1$ grows is relevant

$|\lambda^{(n)}| = 1$ $\phi^{(n)}$ is marginal

Directions that grow are called "relevant"
Directions that shrink are called "irrelevant"
Directions with $|\lambda^{(n)}| = 1$ are called "marginal"

(In practice sometimes one has to be a little more careful and we will return to this... maybe...)

RW via RG

So now lets return to RW... useful to consider slightly more general formulation...

$$p(r) = \frac{1}{(2\pi\sigma_0^2)^{-d/2}} \exp\left(\frac{-|r-r_0|^2}{2\sigma_0^2}\right)$$

} step length is gaussian with mean $\vec{r}_0$ and width σ_0

Step 1: Coarse grain RW by considering n walks together

$$\vec{r}' = \sum_{i=1}^{n} r_i$$

$$p(\vec{r}') = \int dr_1 \cdots dr_n \, \delta(\vec{r} - \sum_{i=1}^{n} r_i) \, p(r_1) \cdots p(r_n)$$

Fourier Transform

$$\int \frac{d^dk}{(2\pi)^d} e^{-ik\cdot r} \prod_{i=1}^{n} \underbrace{\int \left[e^{-ikr_i} p(r_i) \right] dr_i}_{\text{Gaussian integral}}$$

$$\prod_{i=1}^{n} e^{-\frac{|k|^2\sigma^2}{2}} = e^{-\frac{n|k|^2\sigma^2}{2} + ikr_0}$$

$$P(r) = \int \frac{d^d k}{(2\pi)^d} e^{-\left[-n|k|^2 \frac{\sigma_0^2}{2} - ik \cdot r + ikn r_0\right]}$$

$$e^{-\frac{a}{2}\left(x^2 + 2bx + \frac{b^2}{a^2}\right)} \quad e^{\frac{b^2}{2a}}$$

$$\prod_{\alpha=1}^{d} \int \frac{dk_\alpha}{(2\pi)} e^{-n k_\alpha^2 \frac{\sigma_0^2}{2} - ik_\alpha(r_\alpha - n r_{0\alpha})} \Rightarrow \sigma_0^2 \; e^{-\frac{(r - n r_0)^2}{2n\sigma_0^2}}$$

Performind integral over k

$$P(r') = \frac{1}{(2\pi n \sigma_0^2)^{d/2}} \exp -\left[\frac{|r' - n_0 r_0|^2}{2n\sigma_0^2}\right]$$

Unbiased Walk:

Consider first case where $r_0 = 0$ (Unbiased)

Same as original $\Rightarrow$ Coarse grained σ'

$$\sigma' \rightarrow \sqrt{n}\,\sigma_0$$

Step 2 of RG

We want "rescale" our unit of length...

Lets choose RG scheme preserve width

$$\vec{r}'' = \frac{\vec{r}'}{\sqrt{n}} \quad \text{Use bigger ruler}$$

$$P(r'') = \frac{1}{(2\pi \sigma_0^2)^{d/2}} \exp\left[-\frac{(r'')^2}{2\sigma_0^2}\right]$$

Since $d^d r' = n^{d/2} dr''$

So we get $\boxed{\sigma^{(k)} = \sigma^0} \Rightarrow \sigma$ is marginal variable...

Cannot distinguish Renormalized R.W. from origina R.W. $\Rightarrow$ It is, self similar..

Consider

distance coveverd by M steps

$$R_\sigma(M) \sim \sigma M^\nu$$

only dimension

We know that

$$R_\sigma(M) = R_{\sqrt{n}\sigma}\left(\frac{M}{n}\right)$$

$$\sigma M^\nu = \sqrt{n}\,\sigma\left(\frac{M}{n}\right)^\nu \Longrightarrow \boxed{\nu = \frac{1}{2}}$$

So scaling exponent $\nu = 1/2$

We see that $\nu = \frac{1}{D}$ Where D is the fractal dimension...

This RG equation

$\sigma^{(R)} = \sigma^0$ is typical of Gaussian models

However in many ways this is too simple..

Actually, one can really begin to get a feel for RG with just a slightly more difficult example...

~~[illegible]~~ (Biased) R.W.

Imagine now the same process with preferred direction $\vec{r}_0$ .... So that

$$p(\vec{r}) = \frac{1}{(2\pi\sigma^2)^{d/2}} e^{\frac{(\vec{r}-r_0)^2}{2\sigma^2}} \qquad \vec{r}_0 \neq 0$$

Coarse graining this we did this work already

$$p(r') = \frac{1}{(2\pi n\sigma^2)} e^{-\frac{(r'-nr_0)^2}{2n\sigma^2}}$$

Let us coarse grain keep this invariant

Coarse grained parameters $\begin{cases} r_0' = nr_0 \\ \sigma' = \sqrt{n}\,\sigma \end{cases}$

Now let us rescale $\vec{r}' = \sqrt{n}\,\vec{r}'' \Leftrightarrow d\vec{r}'' = \frac{d\vec{r}'}{\sqrt{n}}$

$$P(r'') = \frac{1}{(2\pi\sigma^2)^{d/2}} e^{-\frac{(r''-r_0\sqrt{n})^2}{2\sigma^2}}$$

Then we have that

$$P(r'') = \frac{1}{(2\pi\sigma^{(R)})^2} e^{-\frac{(\vec r'' - r_0^{(R)})}{2\sigma^{(R)2}}}$$

where we have renormalized parameters

$$\sigma^{(R)} = \sigma_0 \Big] \quad \text{doesn't change as before}$$

$$r_0^{(R)} = \sqrt{n}\, r_0 \Big] \quad \text{Gets bigger in magnite ...}$$

Fixed points of this are

$(\sigma, r=0)$ and $(\sigma, r_0 = \infty)$

The first fixed point is "Brownian" motion is pure unbiased RW

The second fixed point note that $R = \sum_{i=1}^{M} \vec r_i$

$$\langle |R|^2 \rangle = \left\langle \sum_{i=1} \vec r_i \cdot \vec r_j \right\rangle$$

$$= \sum_{j=1}^{M}\sum_{i=1}^{M} (\vec r_i - \vec r_0)(\vec r_j - \vec r_0) + 2M \sum_{i=1}^{M} \vec r_i \cdot \vec r_0 - M|\vec r_0|^2$$

$$= M\langle (r_i - r_0)^2 \rangle + 2M^2 |r_0|^2 - M^2 |r_0|^2$$

$$= M(\sigma^2 + M r_0^2) = M\sigma^2 + M^2 r_0^2$$

At large M

$$\sqrt{\langle |R|^2 \rangle} \simeq \sqrt{M^2 r_0^2 + M\sigma_0^2}$$

$$\simeq M r_0$$

So that # of spheres of size a needed to cover walk

$$N(r_0) \sim \frac{M r_0}{a} \sim \frac{1}{a} \quad \text{and } D=1$$

This is just ballistic motion ....

In other words the bias $\vec{r}_0$ is a "relevant" petrubation..

At short distances $\frac{|r|}{\sigma} \ll 1$ the walk looks unbiased (Brownian)

but as one examines the wall at long scales it looks ballistic