Momentum Space RG: ϵ-expansion

We are now in position to do some RG for the ϕ^4 theory...

The key idea is to do this perturbatively by introducing a small parameter $\epsilon = d-4$ (in fractional dimension) ...

This is a "formal" perturbation theory... One can view it as an analytic continuation..

$$\bar{H}^p_\Lambda = u_p \int d^d x \, \phi^p(\vec{x}) \qquad p=4 \text{ is } \phi^4 \text{ theory}$$

The full Hamiltonian is

$$\bar{H}_\Lambda = \bar{H}_{0,\Lambda} + \bar{H}^p_\Lambda$$

where $$\bar{H}_{0,\Lambda} = \frac{1}{2}\int_0^\Lambda \frac{d^d\vec{q}}{(2\pi)^d} (r + c|q|^2)|\phi(\vec{q})|^2 + V_0$$

In momentum space

$$\bar{H}^p_\Lambda = u_p \int_{\vec{q}_1 \cdots \vec{q}_{p-1}} \frac{d^d\vec{q}_1}{(2\pi)^d} \frac{d\vec{q}_2}{(2\pi)^d} \cdots \frac{d\vec{q}_{p-1}}{(2\pi)^d} \, \phi(\vec{q}_1)\,\phi(\vec{q}_2) \cdots \phi(-\vec{q}_1-\vec{q}_2 \cdots -\vec{q}_{p-1})$$

How does U_p scale near the Gaussian fixed point

replace $\phi(\vec{q})$ by $\phi^<(\vec{q})$

Rescale $\vec{q}' = b\vec{q}$ $\qquad$ $\phi^<\left(\frac{\vec{q}'}{b}\right) = \zeta\,\phi'(\vec{q}')$

$$\mathcal{H}^b_\Lambda = U_p \int \frac{d^d q_1'}{(2\pi)^d} \cdots \frac{d^d q_{p-1}'}{(2\pi)^d}\, \zeta^p\, b^{-(p-1)d}\, \phi'(q_1') \cdots \phi'(q_p')\phi(-\vec{q}_1 \cdots -\vec{q}_p$$

$\Rightarrow$

$$U_p' = U_p \zeta^p b^{-(p-1)d} = b^{[-(p-2)d+2p]/2} U_p = b^{\lambda_p} U_p$$

$$\lambda_p = \frac{[2p-(p-2)d]}{2} \quad \text{for } p=1 \ \frac{(d+2)}{2}$$

Now consider $p=4$

$$\lambda_p = \frac{[8-2d]}{2}$$

We see $d_c = 4$ is critical dimenstion

for $d > 4$ $\qquad$ $\lambda_p < 0$ (flows to zero)

$d < 4$ $\qquad$ $\lambda_p > 0$ flows to non-zero

General

$$d > d_c(p) = \frac{2p}{p-2} \quad \text{for } p > 2$$

We could also consider a more general interaction . . .

$$H^4_\lambda = \int d^d\vec{x}_1 \cdots d^d\vec{x}_4 \, U(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4)\,\phi(\vec{x}_1)\phi(x_2)\phi(\vec{x}_3)\phi(\vec{x}_4)$$

In Fourier transfor language we have

$$\tilde{U}(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4) = \int_{\vec{q}_1,\vec{q}_2,\vec{q}_3,} \tilde{U}(\vec{q}_1,\vec{q}_2,\vec{q}_3)(2\pi)^d\,\delta(\vec{q}_1+q_2+\vec{q}_3+\vec{q}_4)$$

Notice we can expand in Taylor series..

$$U(\vec{q}_1,\vec{q}_2,\vec{q}_3) = U + \underbrace{O(\vec{q}_i^{\,2})} \quad \checkmark$$

(From translation symmetry and Isotropic interactions)

$$= U + \underbrace{U_2|q|^2} + \cdots \underbrace{|q|^4}_{b^{-4}} \cdots$$

Notice under rescaling it comes with extra power of b^{-2} when written in terms of $\vec{q}' = b\vec{q}$

So is more irrelevant than U

For this reason we can consider constant U if we are interested in the long distance physics..

The key realization is that since

$$U \to b^{[4-d]}U \quad \text{if} \quad \epsilon = 4-d$$

is small then U can be made sufficiently small to do perturbation theory when ϵ is small enough..

Let us write $\bar{H}_{\Lambda} = \underbrace{\bar{H}_{0,\Lambda}}_{\text{Gaussian}} + \bar{H}^{4}_{\Lambda} \leftarrow \phi^4 \text{ part}$

So let us now do a perturbative calculation of RG in ϵ

As before we write $\phi(\vec{q}) = \phi^{<}(\vec{q}) + \phi^{>}(\vec{q})$

$$\phi^{<}(\vec{q}) = \begin{cases} \phi(\vec{q}) & \text{if } 0 < |q| < \frac{\Lambda}{b} \\ 0 & \text{if } \frac{\Lambda}{b} < |q| < \Lambda \end{cases}$$

$$\phi^{>}(q) = \begin{cases} 0 \\ \phi(\vec{q}) \end{cases}$$

Then we can write

$$\bar{H}_{\Lambda/b} = \int d^d\vec{x} \left[\frac{1}{2} r^{<} (\phi^{<})^2 + \frac{1}{2} c^{<} (\nabla \phi^{<})^2 + u^{<} (\phi^{<})^4\right]$$

where

$$\int D\phi_{<}\, e^{-\bar{H}_{\Lambda/b}[\phi^{<}]} = \int D\phi_{<}\, e^{-\bar{H}^{<}_{0,\Lambda}[\phi^{<}]} \left[\int D\phi^{>}\, e^{-H_{0,\Lambda}[\phi^{>}]}\, e^{-H^{4}_{\Lambda}[\phi^{<}+\phi^{>}]}\right]$$

$\Longrightarrow$

$$e^{-\bar{H}_{\Lambda/b}[\phi^{<}]} = e^{-H^{<}_{0,\Lambda}[\phi^{<}]} \underbrace{\left\langle e^{H^{4}_{\Lambda}[\phi^{<}+\phi^{>}]} \right\rangle_{\phi^{>}}}_{}$$

expectation value of moment generating function

$\Longrightarrow$ Better to work in cumulants

$$e^{\underbrace{\log \langle e^{H^{4}_{\Lambda}[\phi^{<}+\phi^{>}]} \rangle}_{}}$$

cumulant generating function..

This implies that

$$\bar{e}^{H_{\frac{\Lambda}{b}}[\phi^<]} = e^{-H^<_{0,\Lambda}[\phi^<]}\, e^{-\left(\langle \bar{H}^4_\Lambda \rangle + \frac{1}{2}[\langle (\bar{H}^4_\Lambda)^2 \rangle - \langle \bar{H}^4_\Lambda \rangle^2] + \dots\right)}$$

So we just have to calculate cumulants of

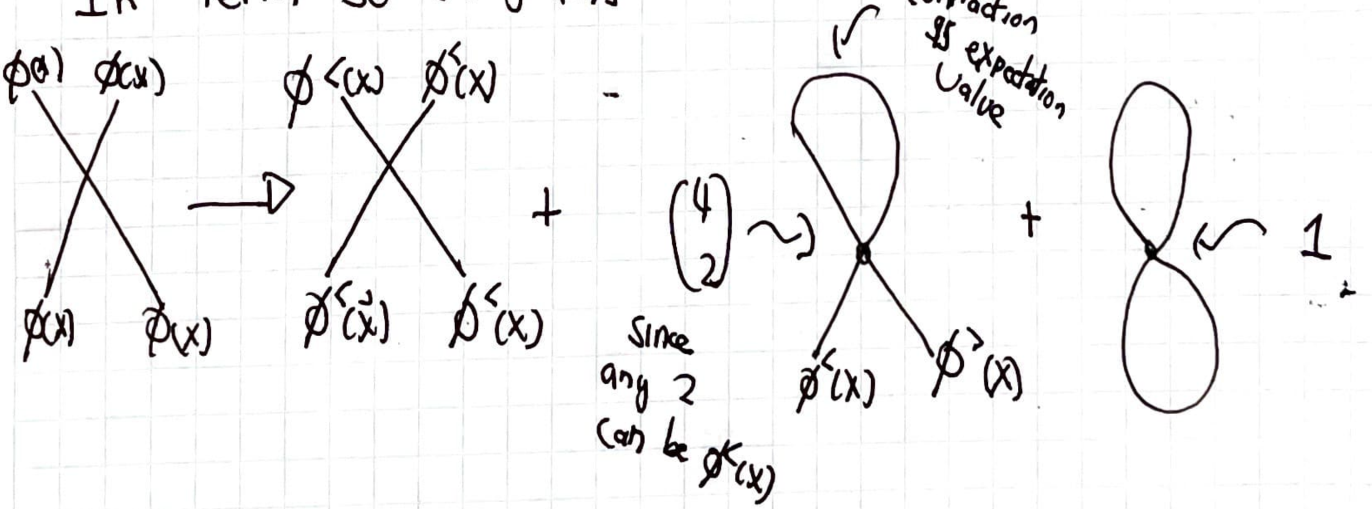
(mean) $\langle \bar{H}^4_\Lambda \rangle_>$

(variance) $\langle (H^4_\Lambda)^2 \rangle_> - \langle \bar{H}^4_\Lambda \rangle^2_>$

So let us conside first term

$$\langle \bar{H}^4_\Lambda \rangle_> \doteq u \int d^dx\, (\phi^<(x))^4 + 6u \int d^dx\, (\phi^<(\vec{x}))^2 \langle (\phi^>(x))^2 \rangle_> + u \int d^dx \langle (\phi^>(\vec{x}))^4 \rangle_>$$

In terms so diagrams



Notice terms proportional to $[\phi^<(x)]^2$

$$\| \qquad \frac{1}{2}\underbrace{[r + 12u \langle \phi^>(q)^2 \rangle_>]}_{r^<} [\phi^<(x)]^2$$

From $H^<_{0,\Lambda}$

⑥

Note that

$H_{0,\Lambda}[\phi^{>}]$ is "Gaussian"

$$= \int d^dx \; \frac{1}{2} r [\phi^{>}]^2 + \frac{1}{2} [\nabla \phi^{>}]^2$$

$$= \int \frac{d^dq}{(2\pi)^d} \frac{1}{2} r [(\phi^{>}(\vec{q}))^2 + |q|^2 [\phi^{>}(q)]^2])$$

$$\int_{\frac{\Lambda}{b}}^{\Lambda} \frac{d^d \cancel{q}}{(2\pi)^d} \frac{1}{2} (r + |q|^2) [\phi^{>}(q)]^2]$$

This implies that

$$\langle (\phi^{>}(\cancel{x}))^2 \rangle_{>} = \int_{\frac{\Lambda}{b}}^{\Lambda} \frac{d^dq}{(2\pi)^d} \frac{1}{r^2 + |q|^2} \sim \int_{\vec{q}}^{>} G_0(q)$$

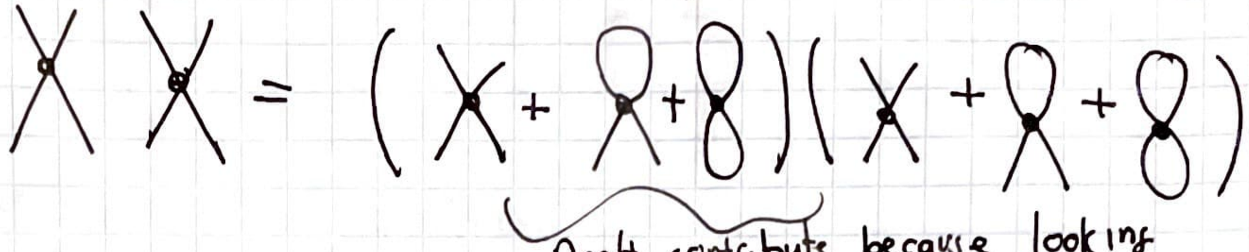
So the renormalized coupling

$$r' = \underbrace{b^{-d} \zeta^2}_{\text{usual scaling}} [r + 12u \int_q^{>} G_0(\vec{q})]$$

Now we can do something similar for u...
Actually need to go to second order

$$\ldots \langle (H_\Lambda^4)^2 \rangle_{>} - \langle H_\Lambda^4 \rangle_{>}^2$$

$\langle (\mathcal{H}^4_\Lambda)^2 \rangle_>$ $\langle (\bar{\mathcal{H}}^4_\Lambda) \rangle_>$ $\langle \bar{H}^4_\Lambda \rangle_\Lambda$



Don't contribute because looking at cumulant

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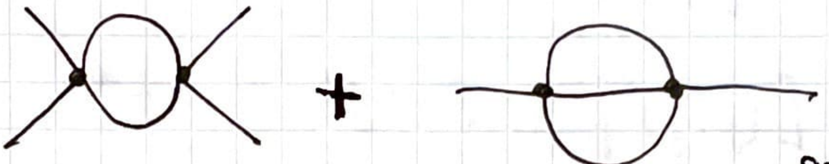
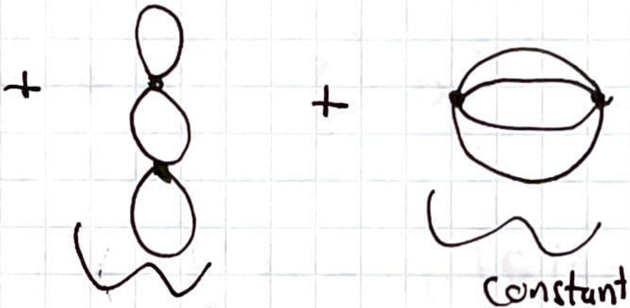


$\underbrace{\binom{4}{2}\binom{4}{2}}_{36}$

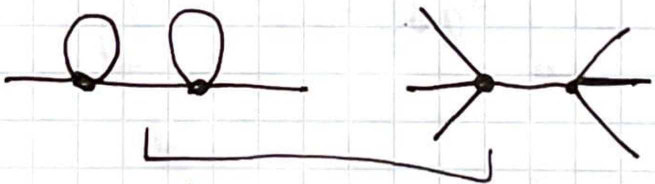
Second order correction to r_0 since proportional to $[\phi^<(x)]^2$



constant

constant

\+



1 particle irreducible do not contribute but irrelevant for our purposes..

$$U'_< = U - 36U^2 \int_q G_0^2(q)_<$$

Two propagators

$$\Rightarrow U' = \underbrace{b^{-3d}\zeta^4}_{\text{Usual scaling}}\left[U - 36U^2 \int_q G_0^2(q)\right]$$

To proceed as usual we can write

$$b = (1 + \delta l)$$

and use RG equaliton

$$r' = b^{-d} \zeta^{2} \left[r + 12 u \int_q^{>} G_0(\tilde{q}) \right]$$

$$u' = b^{-3d} \zeta^{4} \left[u - 36 u^{2} \int_q^{>} G_0^{2}(\tilde{q}) \right]$$

Plugginins in $\zeta^2 = b^{d+2}$.

$\Longrightarrow$

$$r' = b^{2} \left[r + 12 u \int_q^{>} G_0(q) \right]$$

$$u' = \underbrace{b^{-d+4}}_{b^{\epsilon}} \left[u - 36 u^{2} \int_q^{>} G_0(q) \right] .$$

$$\int_{\Lambda(1-dl)}^{\Lambda} d^d q \, G_0(q) = \frac{dl}{\Lambda + r^2}$$

$\Longrightarrow$

$$\boxed{\begin{aligned} \frac{dr}{dl} &= 2r(l) + 12 u \overset{\text{Volume}}{K_d} \frac{u}{1 + r} \\ \frac{du}{dl} &= \epsilon u - 36 u^{2} K_d \frac{u^{2}}{1 + r^{2}} \end{aligned}}$$

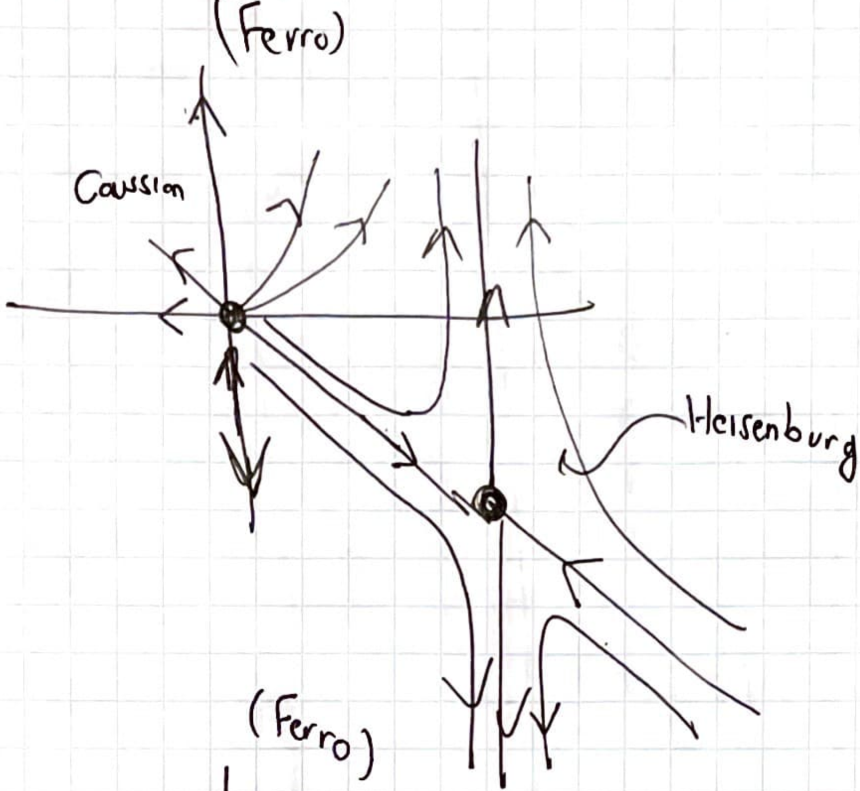
Let us measure units of length in Λ^{-1} so that $\Lambda = 1$

$\Longrightarrow$ Fixed points are $u^* = 0$ $r^* = 0$ (Gaussian)

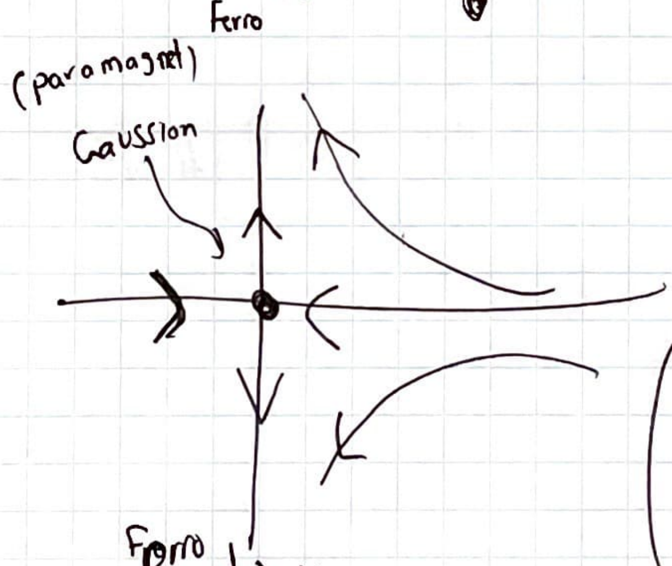
$$u^* = \frac{\epsilon}{36 K_d} + O(\epsilon^2)$$ New fixed point!

(called "Heisenburg Fixed Point") $$r^* = -\frac{1}{6}\epsilon + O(\epsilon^2)$$

This means that the flows look like



$d<4$



$d>4$

We can also calculate Scaling dimensions by linearizing RG equations around u^*, r^* (Heisenburg)

$$\begin{pmatrix} \frac{d\delta r}{d\ell} \\ \frac{d\delta u}{d\ell} \end{pmatrix} = \begin{pmatrix} 2-\frac{1}{3}\epsilon & \frac{12K_d}{1+r^*} \\ 0 & -\epsilon \end{pmatrix} \begin{pmatrix} \delta r \\ \delta u \end{pmatrix}$$

$$\lambda_t = \frac{1}{\nu} = 2 - \frac{\epsilon}{3} \Rightarrow \nu = \frac{3}{6-\epsilon}$$

Not M.F. !!

$$\lambda_u = -\epsilon$$

.... We have found qualitative behavior we wanted..

We can compare for $d=3$ the critical exponents

Exponent	Landau	Ising $O(\epsilon)$	~~A~~Numerical
α	jump	$\frac{\epsilon}{6} = 0.17$	0.110
β	$\frac{1}{2}$	$\frac{1}{2} - \frac{\epsilon}{6} = 0.33$	0.326
γ	1	$1 + \frac{\epsilon}{6} = 1.17$	1.24
δ	3	$3 + \epsilon = 4$	4.79
ν	$\frac{1}{2}$	$\frac{1}{2} + \frac{\epsilon}{12} = 0.58$	0.630
η	0	$O(\epsilon^2)$	0.036

So pretty good!!