PY 895 Homework 4. **Fall 2024** **Due: Tuesday November 26th**

1. Numerical Check of random symmetric equations. Numerically simulate the random differential equations we discussed in class:

$$\lambda x_i + \sum_{j=1}^{N} A_{ij} x_j = b_j \tag{1}$$

where i, j run over $1, \ldots, N$ and $\langle A_{ij} \rangle = \frac{\mu}{N} + \sigma a_{ij}$ with

$$\begin{aligned} \langle a_i j \rangle &= 0 \\ \langle a_{ij} a_{kl} \rangle &= \frac{1}{N} \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right), \end{aligned} \tag{2}$$

and

$$\begin{aligned} \langle b_i \rangle &= \bar{b} \\ \langle (b_i - \bar{b})(b_j - \bar{b}) \rangle &= \sigma_b^2 \delta_{ij}. \end{aligned} \tag{3}$$

(a) Show that the numerics for the distribution for a single x_i averaged over many different realizations of A and b agree with our analytic results.

(b) Now simulate a single very large system. How big do you need to make N before it starts to self-average?

2. Using the zero-temperature cavity method to derive the Marchenko Pastur Distribution

Consider a $N \times M$ dimensional random matrix C whose entries are drawn from a random distribution with means and variances given by

$$\langle C_{i\alpha} \rangle = 0 \tag{4}$$

$$\langle C_{i\alpha} C_{j\beta} \rangle = \frac{\sigma^2}{N} \delta_{ij} \delta_{\alpha\beta} \tag{5}$$

Furthermore define the ration $\gamma = M/N$. Define a Wishart matrix $A = CC^T$. We will calculate the spectrum of A using the zero temperature cavity method.
(a) Define the linear set of equations

$$z u_i = \sum_j A_{ij} u_j + b_i. \tag{6}$$

Argue that we can write the spectrum $\rho_A(x)$ of A in terms of the trace of the susceptibility matrix

$$\nu_{ij} = \frac{\partial u_i}{\partial b_j} \tag{7}$$

as

$$\rho_A(x) = \frac{1}{\pi} Im[\tilde{\nu}(x - i0^+)] \tag{8}$$

where $\tilde{\nu}(z) = \lim_{N\to\infty} \frac{1}{N} \sum_j \nu_{jj}$.
(b) In order to make use of the cavity method and invoke the central limit theorem, we must be able to take averages over the elements $C_{i\alpha}$. To do this, show that we can rewrite these equations as

$$z u_i = \sum_\alpha C_{i\alpha} v_\alpha + a_i \tag{9}$$

$$v_\alpha = \sum_j C_{j\alpha} u_j + b_\alpha. \tag{10}$$

Identify the two mean field variables that we assume are Gaussian under the replica symmetric ansatz.

(c) Define the four susceptibility matrices

$$\nu_{ij}^{(u)} = \frac{\partial u_i}{\partial a_j}, \qquad \nu_{\alpha j}^{(v)} = \frac{\partial v_\alpha}{\partial a_j}$$
$$\chi_{i\beta}^{(u)} = \frac{\partial u_i}{\partial b_\beta}, \qquad \chi_{\alpha\beta}^{(v)} = \frac{\partial v_\alpha}{\partial b_\beta} \tag{11}$$

and show they satisfy the equations

$$\nu^{(u)} = (zI_N - A)^{-1} = \tfrac{1}{Z-A} \tag{12}$$
$$\chi^{(u)} = (zI_N - A)^{-1}C \tag{13}$$
$$\nu^{(v)} = C^T(zI_N - A)^{-1} \tag{14}$$
$$\chi^{(v)} = I_M + C^T(zI_N - A)^{-1}C. \tag{15}$$

(d) Introduce two new variables u_0 and v_0 and an argument similar to class to show that under the assumption of replica symmetry, in the limit where $N, M \to \infty$ and γ held fixed, to leading order in N these new variables satisfy the equations

$$u_0 = \tfrac{\sum_\alpha C_{0\alpha} v_{\alpha/0} + a_0}{z - \sigma^2\gamma\tilde{\chi}} \tag{16}$$
$$v_0 = \tfrac{\sum_j C_{j0} u_{j/0} + b_0}{1 - \sigma^2\tilde{\chi}} \tag{17}$$
$$\tag{18}$$

where $\tilde{\chi} = \frac{1}{M}\sum_\alpha \chi_{\alpha\alpha}^{(v)}$ and $\tilde{\nu} = \frac{1}{N}\sum_j \nu_{jj}^{(v)}$.

(e) Explain in the calculation above why you have to introduce two new variables and how it relates to the number of mean-fields in the problem. In particular, what goes wrong if you do not introduce both variables.

(f) Use the equations above to derive the self-consistency equations

$$\tilde{\nu} = \tfrac{1}{z - \sigma^2\gamma\tilde{\chi}}$$
$$\tilde{\chi} = \tfrac{1}{1 - \sigma^2\tilde{\chi}} \tag{19}$$

(g) Use this in conjunction with part (a) to show that

$$\rho_A(x) = \begin{cases} (1-\gamma^{-1})\delta(x) + \frac{1}{2\pi x\sigma^2}\sqrt{(x - x_{min})(x_{max} - x)}, & \text{if } \gamma \geq 1 \\ \frac{1}{2\pi x\sigma^2}\sqrt{(x - x_{min})(x_{max} - x)}, & \text{if } \gamma \leq 1 \end{cases}, \tag{20}$$

where

$$x_{min} = (1+\gamma)\sigma^2 - 2\sqrt{\gamma}\sigma^2$$
$$x_{max} = (1+\gamma)\sigma^2 + 2\sqrt{\gamma}\sigma^2 \tag{21}$$

This is called the Marchenko-Pastur distribution.

(h) Check this expression numerically by drawing random matrices of different sizes $M \times N$. For what values of M and N does this distribution give a reasonable description.

3. Completing derivation of RS breaking line. In this problem, we will fill in the details of the RS breaking in the generalized Lotka-Volterra model that was discussed in class. Our starting point are the self-consistence equation for a new species N_0 introduced in the ecosystem. Please see Chapter 8 of these Les Houches Lectures `https://arxiv.org/abs/2403.05497`. In particular our starting point is Eq. 77. in these notes We will consider the special case of this equation where $\sigma_r^2 = 0$.

$$N_0 = \max\left[0, \frac{r - \mu\langle N\rangle + \sqrt{\sigma^2\langle N^2\rangle} z_N}{1 - \rho\sigma^2\nu}\right] \tag{22}$$

(b) Consider perturbing all the non-extinct species from there steady state values $\vec{N}^* \to \vec{N}^* + \epsilon\vec{\eta}$ where η is a random vector. Show that under the assumption of Replica Symmetry, the square of the expectation value of the derivative $\langle\left(\frac{\partial N_0}{\partial \epsilon}\right)^2\rangle_+$ diverges exactly when diverges precisely when

$$\nu_c = \frac{1}{\sigma_c^2(1+\rho)}. \tag{23}$$

To do so, make use of Eq. 83 for ν

$$\nu = \frac{\phi_N}{1 - \rho\sigma^2\nu} \tag{24}$$

(c) For the special case where $\sigma_r^2 = 0$ we can solve explicitly for the critical ν. To do so, we must make use of the identity

$$w_2(\Delta) = w_0(\Delta) + \Delta, \tag{25}$$

where $w_j(\Delta)$ is defined in Eq. 82 in the Les Houches Lectures

$$w_j(\Delta) = \frac{1}{\sqrt{2\pi}}\int_{-\Delta}^{\infty} dz e^{\frac{-z^2}{2}}(z+\Delta)^j. \tag{26}$$

Prove this identity.

(d) Using the self-consistency equations Eq. 83-85 in Les Houches lectures, the identity above, and substituting in the critical relationship for RS breaking, $\nu_c = \frac{1}{\sigma_c^2(1+\rho)}$, show that when $\sigma_r^2 = 0$ the criteria for RS breaking reduces to

$$\frac{1}{\sigma_c^2} = \frac{1+\rho}{\sqrt{2}}. \tag{27}$$