PY 895 Homework 3. **Fall 2024** **Due: Tuesday November 5th**

1. Practice with scaling

Consider a free energy for a complex scalar field ψ coupled to a gauge field A_i,

$$F(\psi, A_i) = \int dx^d \frac{1}{4} F_{ij} F^{ij} + |\partial_i \psi - ieA_i\psi|^2 + \mu^2 |\psi|^2 \tag{1}$$

with $F_{ij} = \partial_i A_j - \partial_j A_i$.

(a) What is the critical dimension d_c such that the coupling between the scalar and gauge field is relevant for $d < d_c$ and irrelevant for $d > d_c$? Speculate on what you think happens when $d = d_c$.

(b) If one were to add a coupling $g|\psi|^4$ to this free energy, explain how the coupling g scales and when it is relevant and irrelevant?

2. Practice with RG using epsilon expansion I. Derive the RG equation to first order in $\epsilon = d - 4$ for the O_n vector model (which generalizes the Ising model so that there are now n scalar fields). The free energy for this model is given by

$$F = \int dx \left[\sum_i \frac{r}{2} \phi_i^2 + \frac{1}{2} (\nabla \phi_i)^2 \right] + u(\sum_i \phi_i^2)^2, \tag{2}$$

where $i = 1, \ldots, n$.
(a) Show that the RG equations take the form

$$\begin{aligned} \frac{dr}{dl} &= 2r + 4K_d(n+2)\tfrac{u}{1+r} \\ \frac{du}{dl} &= \epsilon u - 4K_d(n+8)\tfrac{u^2}{(1+r)^2}, \end{aligned} \tag{3}$$

where K_d angular integral in spherical coordinated in d-dimensions.
(b) Show that uthe there are two fixed points, the Gaussian fixed point with $r = u = 0$ and a new Wilson-Fisher fixed point with

$$\begin{aligned} u^* &= \tfrac{\epsilon}{4(n+8)K_d} + O(\epsilon^2) \\ r^* &= -\tfrac{1}{2}\tfrac{n+2}{n+8}\epsilon + O(\epsilon^2) \end{aligned} \tag{4}$$

and show that the scaling dimensions of the two relevant fields are

$$\begin{aligned} \lambda_t &= 2 - \frac{n+2}{n+8}\epsilon \\ \lambda_u &= -\epsilon \end{aligned} \tag{5}$$

(c) Draw a digram of the RG flows for this system.

(d) Calculate the critical exponents for this system.

3. Effect on anisotropy

Consider a system describe by two coupled Ising order parameters with the Landau free energy

$$f = \frac{1}{2} r(\phi_1^2 + \phi_2^2) - \frac{1}{2} g(\phi_1^2 - \phi_2^2) + u(\phi_1^2 + \phi_2^2)^2. \tag{6}$$

Notice that $g \neq 0$ corresponds to field that breaks vector symmetry of $\phi = (\phi_1, \phi_2)$.
(a) Argue that for small u, the Mean Field phase diagram for this system in the r, g plane takes the form shown in the Figure below There is a first order line along $g = 0$, $r < 0$ separating the phase with $\phi_1 \neq 0$ and $\phi_2 = 0$ from the

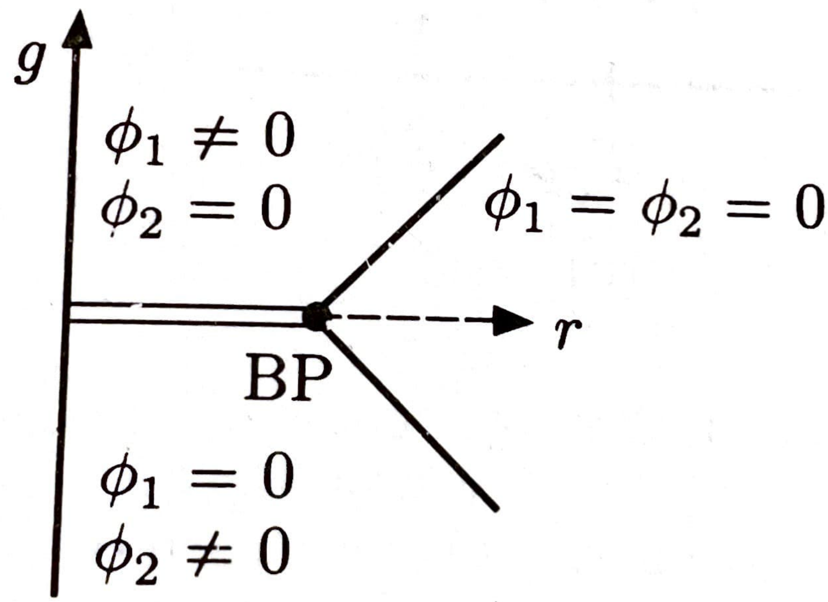


FIG. 1: Mean-field phase diagram

phase where $\phi_1 = 0$ and $\phi_2 \neq 0$. These two distinct second order phase transitions meet at the point $r = 0$ and $g = 0$ which is the bicritical point.
(b) Now consider the generalized free energy

$$F = \int dx \left[\sum_{i=1}^{2} \frac{r_i}{2}\phi_i^2 + \frac{1}{2}(\nabla\phi_i)^2 \right] + u(\sum_{i=1}^{2}\phi_i^2)^2, \tag{7}$$

where $r_1 = r + g$ and $r_2 = r - g$. Show that to leading order in ϵ that one can write the RG equations for the r_1 and r_2 as

$$\begin{aligned} \frac{dr_1}{dl} &= 2r_1 + 4K_d u\left(\frac{3}{(1+r_1)} + \frac{1}{(1+r_2)}\right) \\ \frac{dr_2}{dl} &= 2r_2 + 4K_d u\left(\frac{3}{(1+r_2)} + \frac{1}{(1+r_1)}\right) \end{aligned} \tag{8}$$

(c) Argue (without calculating) that we still expect there be a non-trivial Wilson-Fisher fixed point with $r_1 = r_1^*$, $r_2 = r_2^*$ and $u = u^*$ where $r_1^*, r_2^*, andu^*$ are all order ϵ.

(d) Let us linearize the equations above around this fixed point, so that $r_1 = r_1^* + \delta r_1$, $r_2 = r_2^* + \delta r_2$, and $u = u^* + \delta u$. Show that if we define $\delta r = \delta r_1 + \delta r_2$ and $\delta g = \delta r_1 - \delta r_2$ that linearized RG equations (ignoring terms $O(\epsilon^2)$) become

$$\begin{aligned} \frac{d\delta r}{dl} &= [2 - 16K_d u^*]\delta r \\ \frac{d\delta Q}{dl} &= [2 - 4K_d u^*]\delta Q \end{aligned} \tag{9}$$

(e) Use the equation above to calculate scaling exponent for the anisotropy coupling g.

4. Tensor network renormalization. [Optional]

Consider the nearest-neighbor Ising model on the triangular lattice. The reference for the strategy we will follow here is by Levin and Nave (PRL 2007). Please look and read this The goal is to work through the arguments in this paper.

(a) Show that the partition function may be written as the contraction of a tensor network:

$$Z = Tre^{-\beta H} = trTTTTTTT \cdots = \sum_{ijklmno\cdots} T_{ijk}T_{klm}T_{mno}\cdots \tag{10}$$

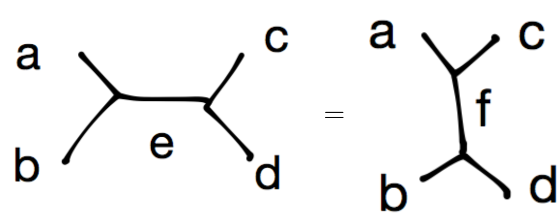
where the tensors T_{ijk} are 3-index objects (tensors) which depend on the couplings, and which are associated with sites of the dual honeycomb lattice. They have one index for each of the incident edges of the honeycomb lattice. Find a set of $T_{ijk}, ijk \cdots = 0, 1$ which makes this equation true, for h = 0.

(b) [slightly harder] Find a set of T s which works for nonzero h.

(c) [slightly harder still] Once we've written Z in this form, we can do a coarse- graining procedure in two steps. First consider a pair of neighboring honeycomb lattice sites, associated with two tensors $\sum_e T_{abe}T_{ecd}$. Regard this object as a $D^2 \times D^2$ matrix with block indices ac and bd. By doing a singular-value decomposition of this matrix, rewrite the product as:

$$\sum_e T_{abe}T_{ecd} \equiv \sum_f S_{acf}S_{fbd} \tag{11}$$

In diagrams, this looks like:



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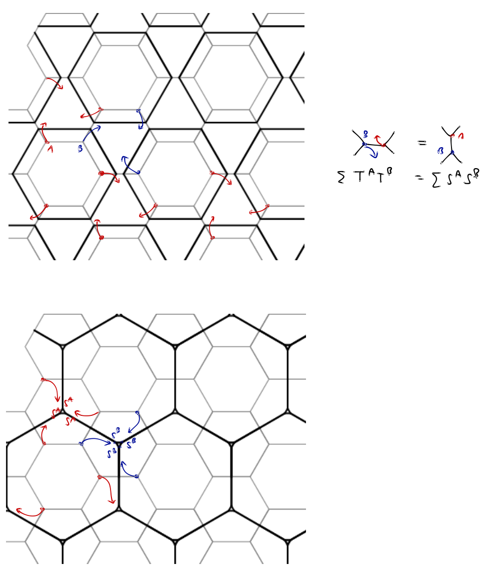
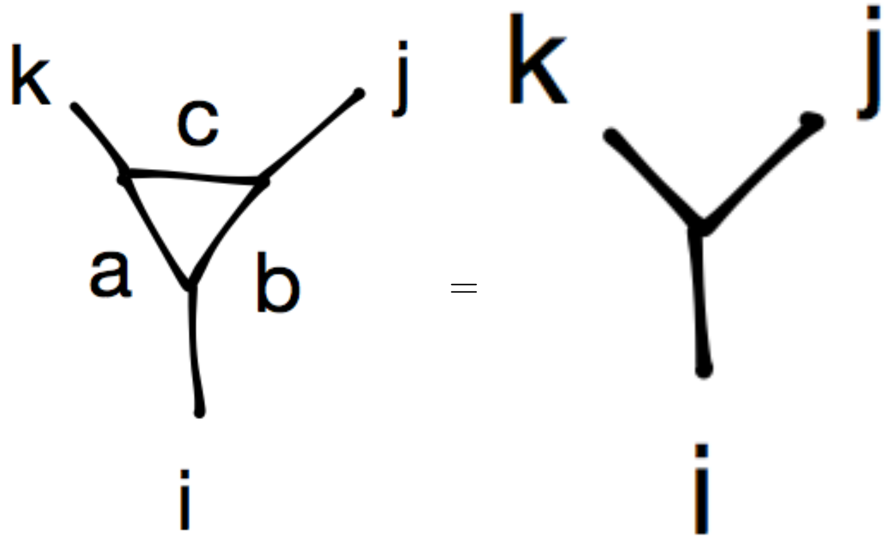


FIG. 2: Tensor RG scheme

We are left with triangles of Ss. The second step of the coarse-graining scheme is to define a new T by

$$\sum_{abc} S_{kac} S_{jcb} S_{iab} = T'_{kij} \tag{12}$$

or in pictures by: This gives back an Ising model on the triangular lattice with a larger lattice spacing.



(d) [Optional] Implement this RG scheme numerically. Notice that the approximation comes in when we throw away singular values in step 1 (if we do not, the range of indices of the tensors (called the bond dimension) must grow with the number of steps). Compute the magnetization as a function of temperature.