PY 895 Homework 2. Fall 2024 Due: Thursday October 1st

1. Flory's argument for scaling exponent of polymer with excluded volume. In this HW problem, we will reproduce Flory's beautiful mean-field argument for calculating the critical exponent ν of the mean displacement $R \sim N^{\nu}$ for a a polymer with excluded volume in d-dimensions. Note that for a sufficiently thin polymer we expect this to be the same as a self-avoiding random walk by universality.

(a) Argue that for a polymer of length R, the mean monomer concentration is $\langle c \rangle = N/R^d$.

(b) Argue that there is an energy cost ϵ for self-interactions so that the repulsive energy per unit volume, $f_{repulsive}$, is given by

$$f_{repulsive} = \epsilon v_d c^2, \tag{1}$$

where v_d the volume associated with a monomer. Use a mean field approximation to argue that the total repulsive interaction energy is

$$F_{repulsive} = \epsilon v_d \frac{N^2}{R^d}. \tag{2}$$

This interaction energy tends to expand the polymer. However, entropic effects tend to balance this expansion.
(c) To calculate the entropic calculation, we model the polymer as an ideal chain (random walker with no interactions). Show that the probability that polymer of N monomers has length R is

$$p(N,R) = \frac{1}{\sqrt{2\pi N b^2}^d} e^{\frac{-dR^2}{2Nb^2}}, \tag{3}$$

where b is the length of a monomer (we did this calculation in class). Use this to show that the to leading order in N the R-dependent portion of the total free energy is

$$F_{Total} \sim \epsilon v_d \frac{N^2}{R^d} + k_B T \frac{dR^2}{2Nb^2} \tag{4}$$

(e) Show that the total free energy has a minimum at $R = R_F$ and use this to show that $R_F \sim N^{\nu}$ with $\nu = 3/(d+2)$.

2. Site percolation on square lattice. Consider site percolation on a two dimensional square lattice where the probability to occupy a site denoted by p. Furthermore, assume that near the percolation transition, the correlation length ξ diverges as

$$\xi = |p - p_c|^{-\nu}. \tag{5}$$

(a) Construct an RG transformation for this system with a scale factor of $b = 2$ with a rule that the coarse grained lattice site is occupied if the cell is spanned from left to right *or* top to bottom. Calculate the critical point p_c and the critical exponent ν. (Hint: to calculate ν linearize your RG transformation).

(b) Repeat the calculation with an RG transformation for a system with $b = 2$ with a rule that the coarse grained lattice site is occupied if the cell is spanned from left to right *and* top to bottom.

(c) Compare your answers to (a) and (b) and give an explanation for what you see.

3. Filling in details of the Feigenbaum calculation. Let us denote our 1-dimensional map $f(x, C)$. As usual, we assume that $f(x, C)$ had one maximum at $x = 1/2$ with $f(1/2) = C$ and and $f(0) = f(1) = 0$. Let us define a family of universal functions $g_r(x)$ of the form

$$g_r(x) = \lim_{n\to\infty} (-\alpha)^n f^{(2^n)}(\frac{x}{\alpha^n}, \lambda_{n+1}), \tag{6}$$

where $f^{(j)}$ denotes f composed j times and λ_n is defined by the equation

$$f^{(2^n)}(\frac{1}{2}, \lambda_n) = \frac{1}{2}. \tag{7}$$

(a) Show that $g_{r-1}(x) = -\alpha g_r(g_r(\frac{x}{\alpha}))$.

(b) Show that if we define the universal function by the equation

$$g(x) = -\alpha g(g(\frac{x}{\alpha})), \tag{8}$$

then if $g(x)$ is a solution to this equation so is $g_\mu(x) = \mu g(x/\mu)$.

4. Simulation of period doubling to chaos. Here we will simulate the period doubling to chaos for your favorite 1-d map. Produce a bifurcation diagram and identify the critical parameter at which chaos appears. This is a nice implementation for logistic map that can be easily modified: https://github.com/gboeing/pynamical. Feel free to use it or write your own code.