Cavity Method as Algorithm

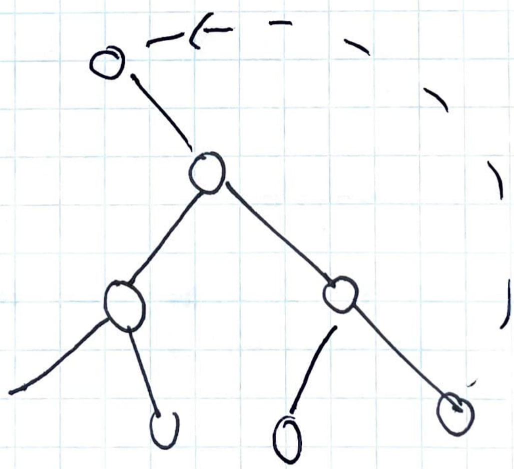
We saw how we can derive self-consistency equations for spin system states...

Instead of solving for statistics of $\{h_i\}$, $\{S_i\}$ in the form of $p\{h_i\}$ and $p\{S_i\}$ averaged over disorder, we would like to calculate things for a single realization of $\{S_i\}$...

Can we find actual "ground state", calculate "partition function", or "Free Energy"...

We would like an algrothmic way of calculating all this,..

The key is that in "high-dimensions", all graphs look locally like trees



The probability of "loop" falls off as $\frac{1}{Z}$ (where Z is number of neighbors...)

So in infinite dimension all things are locally trees...

It is actually interesting to present B.P. for a Factor graph

Consider a partition function and energy . .

It can often be written in terms of sum of local variables . . .

S.K. in one dimensions . . $E = \sum_i J_{ij} S_i S_{i+1}$

$e^{\beta J_{i+2,i+3} S_{i+2} S_{i+3}}$

$e^{\beta J_{i,i+1} S_i S_{i+1}}$

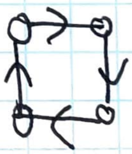
$e^{\beta J_{i+1,i+2} S_{i+1} S_{i+2}}$ } Here all factors involve two variables (spins)

$$E = \sum_{a \in \text{Factors}} E_a(\{S_j\}_{j \in \partial a})$$

same

$$Z = \prod_a \psi_a(S_{\partial a})$$

However in gauge theory energy depends on loops

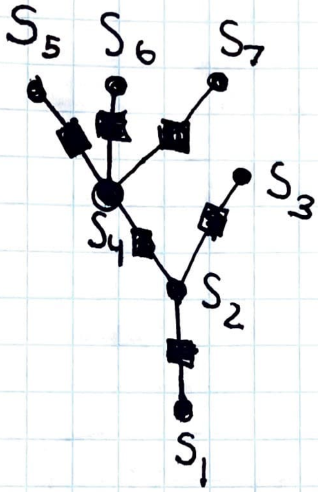


$\prod_{i \in \text{Loop}} S_i$ } Then one factor for each loop.

This is useful because it allows us to think about a bunch of problems statistically from computer science and information theory . . .

Useful to start with "Ising / SK" Model on graph before generalizing to general factor graphs...

So factor graph with "2-body" interactions...

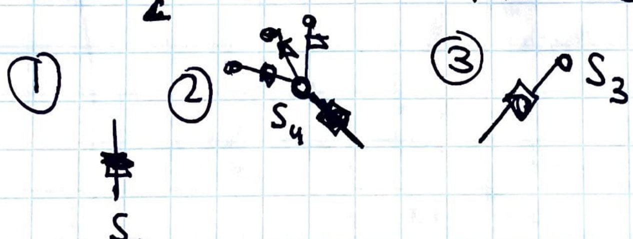


$$Z = \underbrace{\psi_{54}}_{e^{\beta J_{45} S_5 S_4} \cdots}(S_5, S_4)\, \psi_{64}(S_6, S_4)\, \psi_{74}(S_7, S_4) \cdot \psi_{42}(S_4, S_2)\, \psi_{32}(S_3, S_2)\, \psi_{21}(S_2, S_1)$$

More generally

$$Z = \prod_{\langle kj \rangle} \psi_{kj}(S_k, S_j)$$

So now notice that ~~if~~ we imagine removing a node S_2... Notic the 3 "branches" corresponding to



become independent...

The "local field" at spin 4 is just the sum of fields due to each of those branches..

Equivalently, we can see that partition function must factorize... Let us define $Z_{j \to c}(S_j)$ as the partition function for the subtree rooted at c, excluding branch j with fixed value of S_4 (for example $S_4 = +1$ or -1 for spins)

For example, for the figure

$$Z_{2\to 1}(S_2) = \sum_{S_3,S_4,S_5,S_6,S_7} \psi_{23}(S_3,S_2)\,\psi_{42}(S_4,S_2)\,\psi_{54}(S_5,S_4)\,\psi_{64}(S_6,S_4)\,\psi_{74}(S_7,S_4)$$

$$= \sum_{\sigma_3\,\sigma_4} Z_{3\to 2}(\sigma_3)\,Z_{4\to 2}(\sigma_4)\,\psi_{42}(\sigma_4,\sigma_2)\,\psi_{32}(\sigma_3,\sigma_2)$$

It's cleary we can write the Cavity equations as

$$\begin{cases} Z_{i\to j}(S_i) = \prod\limits_{\underbrace{k\in\partial i/j}_{\text{all neighbors of } i \text{ except } j}} Z_{k\to i}(S_k)\,\psi_{ik}(S_i,S_k) \\ Z_i(S_i) = \prod\limits_{k\in\partial i} \left[\sum\limits_{S_k} Z_{k\to i}(S_k)\,\psi_{ik}(S_k,S_i)\right] \end{cases}$$

This is like averaging over local field removing i

These are the B.P. equations

$\Rightarrow$ Cavity $\Rightarrow$ Assume local tree...

Actually often useful to "normalize" these

$$\eta_{i\to j}(S_i) = \frac{Z_{i\to j}(S_i)}{\sum_{S_i'} Z_{i\to j}(S_i')} \qquad \eta_i(\sigma_i) = \frac{Z_i(S_i)}{\sum_i Z_i(S_i)}$$

Which gives

$$\eta_{i\to j}(S_i) = \frac{1}{Z_{i\to j}} \prod_{k\in\partial i\setminus j} \left(\sum_{S_k} \eta_{k\to i}(S_k)\,\psi_{ik}(S_i,S_k) \right)$$

Requiring $\sum_{S_i} \eta_{i\to j}(S_i) = 1$ gives

$$Z_{i\to j} = \sum_{S_i} \prod_{k\in\partial i\setminus j} \left[\sum_{S_k} \eta_{k\to i}(S_k)\,\psi_{ik}(S_i,S_k) \right]$$

And

$$\eta_i(S_i) = \frac{1}{Z_i} \prod_{j\in\partial i} \sum_{S_j} \eta_{j\to i}(S_j)\,\psi_{ji}(S_j,S_i)$$

$$\sum_{S_i} \eta_i(S_i) = 1 \Rightarrow Z_i = \sum_{S_i} \left[\prod_{j\in\partial i} \sum_{S_j} \eta_{j\to i}(S_j)\,\psi_{ij}(S_i,S_j) \right]$$

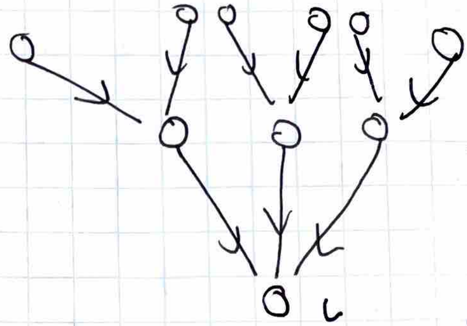
So we now must relate this to "physical quantities" like Free Energy, Magnetizations, ect... So how do we do that?

The magnetization is easy?

$$M_i = \langle S_i \rangle = \sum_{S_i} S_i\,\eta_i(S_i)$$

How do we find the Partition function and Free Energy of these quantities . .

Notice we can write partition function as product of partion functions for trees



$$Z = Z_i \underbrace{\prod_{j \in \partial i} Z_{j\to i} \prod_{k \in \partial j \setminus i} Z_{k \to j} \cdots\cdots}$$

Recursively from leaf..

So actually easier to write in terms of

$$Z_{ij} = \sum_{s_i\, s_j} \eta_{j\to i}(s_i)\, \eta_{i \to j}(s_j)\, \psi_{ij}(s_i, s_j)$$

For neighbors...

Notice

$$Z_{i\to j} = \frac{Z_i}{Z_{ij}}$$

$$Z_{j \to i} = \frac{Z_j}{Z_{ij}}$$

So that

$$Z = \frac{\prod_i Z_i}{\prod_{\langle ij \rangle} Z_{ij}}$$

$$\Longrightarrow \quad F = -T \log Z = \sum f_i - \sum_{\langle ij \rangle} f_{ij}$$

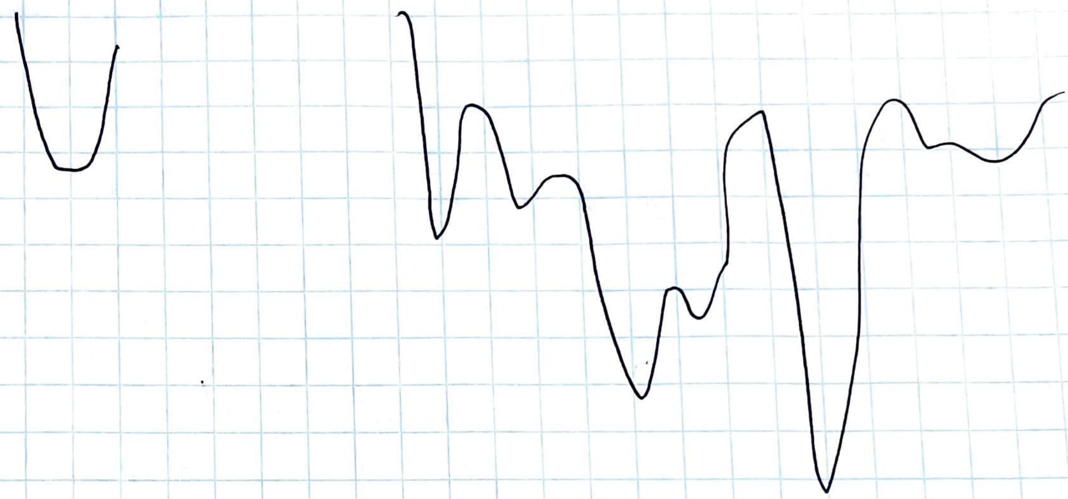
$f_i = -T \log Z_i$ } assuming spins are indep

$f_{ij} = -T \log Z_{ij}$ } corrections due to correlations

What is relation to glasses?

Generally, expect 1 solution when system is not glassy and many solutions when system is glassy

"Landscape":



This can and was made rigourous by Montanari + Dembo (2008)

- Can easily generalize to arbitrary factor graphs...
- Also good approximation even when loops...
  "Loopy Belief Propagation"

$\Rightarrow$ See Chapter 14 on Montanari for more details..