Lecture Phenomenology of the SK Model

We will be interested in studying the cannonical model of a mean-field spin glass...

The Sherrington-Kirkpatrick Model..

$$H = -\sum_{i<j} J_{ij} S_i S_j - h \sum S_i$$

(If on lattice in finite dimensions, called Edward Anderson model)

We assume that the couplings are drawn for distribution

$$P(J_{ij}) = \frac{1}{J}\sqrt{\frac{N}{2\pi}}\, e^{-\frac{N}{2J^2}\left(J_{ij}-\frac{J_0}{N}\right)^2}.$$

As before we will choose the mean and variance are proportional to $1/N$

$$[J_{ij}] = \frac{J_0}{N} \qquad [(\delta J_{ij})^2] = \frac{J^2}{N}$$

So what is the basic phemomendgy we would like to capture...

In the absence of disorder we know there are two phases a <u>paramagentic phase</u> and <u>ferromagnetic phase</u>.

As we will see in the SK model there are "additonal phases"

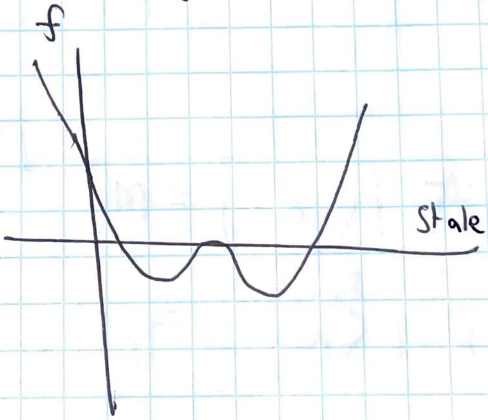
$\Rightarrow$ "a spin glass phase"

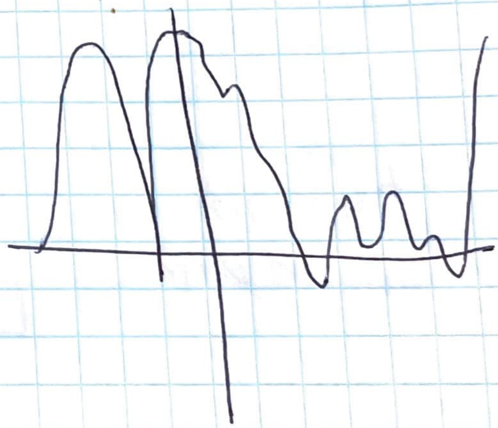
and a mixed phase where R.S. is broken..

To talk about phases, we need to define order parameters.. One is obvious $\langle m \rangle$ (usual magnetization)

The point of a spin glass is actually there are many valleys....

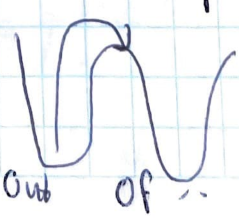
In a ferromagnetic system





In the infinite size limit $N \to \infty$, we can-define the idea of a "pure state"

States that you cannot tunnel out of..



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So what order parameters should we use..

Well the first is usual magnetization but averaged over disorder

$m = [\langle S_i \rangle]$ (dominated by value in valleys)

The second order parameter is (E.A. price)

$q = [\langle S_i \rangle^2]$

So what do we expect.. In paramagnet expectation

Paramagnet

$m = [\langle S_i \rangle] = 0$
$q = [\langle S_i \rangle^2] = 0$ } obviously since spin fluctuates..

Ferromagnet

$m = [\langle S_i \rangle] \neq 0$
$q = [\langle S_i \rangle^2] \neq 0$ } In ferromagnet spins are frozen

Spin Glass

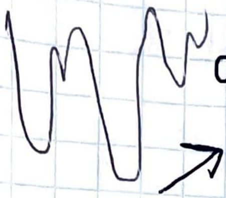
$m = [\langle S_i \rangle] = 0$ → while $\langle S_i \rangle \neq 0$ in any valley across valleys average random to zero

$q = [\langle S_i \rangle^2] \neq 0$

since nonzero and $(\pm 1)^2 \approx 1$ so does not average to zero..

Actually can be a little bit better defined..

We can the magnetization in the "pure state" / valley



$$q_{EA} = \left[\sum_a P_a (m^a)^2 \right] = \sum_a P_a \frac{1}{N} \sum_i (m_i^a)^2$$

Endward Anderson Order paramters

P_a is probability of being in pure state

Another order parameter overlap $\bar{q}$

$$\bar{q} = \left[\left(\sum_a P_a m_i^a \right)^2 \right] = \left[\sum_{ab} P_a P_b m_i^a m_i^b \right]$$

Connected

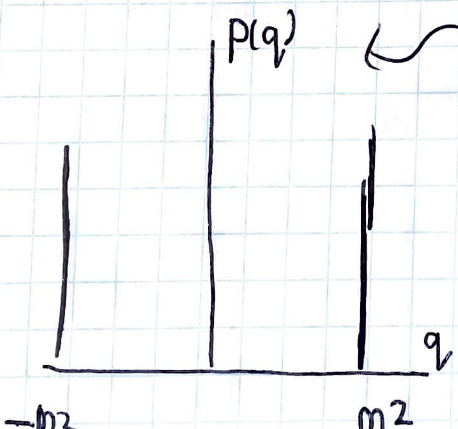
$$= \frac{1}{N} \left[\sum_{ab} P_a P_b \sum_i m_i^a m_i^b \right]$$

writing $m_i = \sum_a P_a m_i^a$ as

$$\bar{q} = [m_i^2] = [\langle S_i \rangle^2]$$

expectation in valley..

What is probability of q



multivalley structure...

Finite Temperature Cavity Method

Let us first start by finding the R.S. solution for the S.K. model.. We will then see how this naturally motivates a set of algorithms "Belief Propagation"

The crucial differences from before are that:

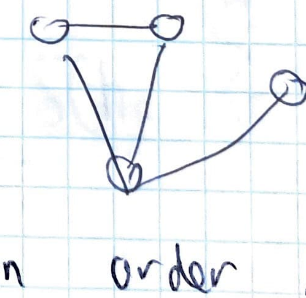
1. Before we were working at "zero-temperature" (i.e no stochasticity), no we have to think about averaging over <u>thermal fluctuations</u>

2. Whereas before we were working directly with the "Equations of Motion", we will deal with <u>partition functions + free energies</u>

As before we imagine taking a system of N spins, and adding a new spin S_0..
For simplicity we consider zero-external field case

$$H_{N+1} = H_N - h_0 S_0$$

$$h_0 = \sum_{j=1}^{N} J_{0j} S_j$$



Again notice that this is an order $1/N$ perturbation to the Free-energy and physics.

As before we shall assume

$$P(J_{ij}) = \frac{1}{J}\sqrt{\frac{N}{2\pi}}\, e^{\frac{-N\left(J_{ij}-\frac{J_0}{N}\right)^2}{2J^2}}$$

We can now write the probability of a state $P\left(\{S_i\}_{i=0}^{N}\right)$ in the $(N+1)$-spin system as

$$P(\{S_i\}_{i=0}^{N}) = \frac{1}{Z_{N+1}} e^{-\beta H^{N+1}(\{S_i\}_{i=0}^{N})}$$

and

$$Z_{n+1} = \mathrm{Tr}_{\{S_i\}_0^N}\, e^{-\beta H^{N+1}}$$

We Know that we can relate this to probabilities of smaller system by introducing the mean field "h_0" ... In fact, we will calculate the <u>marginal probability</u> of (h_0, S_0)

$$P(h_0, S_0) = \frac{1}{Z_{n+1}} \mathrm{Tr}_{\{S_i\}_{i=1}^{N}} \left[\delta\left(h_0 - \sum_{j=1}^{N} J_{0j} S_j\right) e^{-\beta H^{(N+}}\right.$$

Now we can also calculate the marginal probability of the field...

$$P^{N+1}(h_0, S_0) = \frac{1}{Z_{N+1}} \mathrm{Tr}_{\{S_i\}_{i=1}^N}\left[e^{\beta h_0 S_0} \delta\left(h_0 - \sum_{j=1}^{N} J_{0j} S_j\right) e^{-\beta H^{(N)}} \right]$$ (7)

$$\Updownarrow$$

$$P^{N+1}(h_0) = \frac{1}{Z_{N+1}} \mathrm{Tr}_{\{S_i\}_{i=1}^N}\left[\sum_{S_0=\pm 1} e^{\beta h_0 S_0} \delta\left(h_0 - \sum_{j=1}^{N} J_{0j} S_j\right) e^{-\beta H^{(N)}} \right]$$

This almost looks like $P^N(h)$ ← (wrong normalization prefactor and

Define

$$P^N(h_0) = \frac{1}{Z_N} \mathrm{Tr}_{\{S_i\}_{i=1}^N}\left[\delta\left(h_0 - \sum_{j=1}^{N} J_{0j} S_j\right) e^{-\beta H^{(N)}} \right]$$

So we see that

$$P^{N+1}(h_0, S_0) = \frac{Z_N}{Z_{N+1}} e^{\beta h_0 S_0} P^N(h_0) \equiv \frac{1}{\xi} e^{\beta h_0 S_0} P^N(h_0)$$

Notice

$$Z_{N+1} = \mathrm{Tr}_{\{S_i\}_{i=0}^N} e^{-\beta h_0 S_0} e^{-\beta H^{(N)}}$$

$$= \mathrm{Tr}_{\{S\}_{i=1}^N} \cosh \beta h_0 \, e^{-\beta H^{(N)}}$$

$$= \mathrm{Tr}_{\{S_i\}_{i=1}^N} \cos \beta h_0 \, P^N(\{S\}) \cdot Z_N$$

$\Longrightarrow$

$$\xi = \frac{Z_{N+1}}{Z_N} = \langle 2 \cosh \beta h_0 \rangle_N$$ ⇝ average w.r.t. to

$$P^N(\{S_i\}) = \frac{e^{-\beta H^{(N)}}}{Z^N} \ldots$$

So let us now calculate the average magnetization of new spin

$$\langle S_0 \rangle_{N+1} = \mathrm{Tr}_{S_0 = \pm 1} S_0 \int P^{N+1}(h_0, S_0)\, dh_0$$

$$= \mathrm{Tr}_{S_0 \pm 1} \int dh_0\, S_0 \frac{e^{+\beta h_0 S_0}}{\xi} P^N(h_0)$$

$$\langle S_0 \rangle_{N+1} = \frac{\langle \sinh \beta h_0 \rangle_N}{\langle \cosh \beta h_0 \rangle_N}$$

We can also calculate the expectation value

$$\langle h_0 \rangle_{N+1} = \mathrm{Tr}_{S_0} \int dh_0\, h_0\, P^{N+1}(h_0, S_0)$$

$$= \mathrm{Tr}_{S_0} \int dh_0\, \frac{e^{\beta h_0 S_0} h_0}{\xi} P^N(h_0)$$

$$= \frac{\int dh_0\, (2\cosh \beta h_0) h_0\, P^N(h_0)}{\langle 2\cosh \beta h_0 \rangle_N}$$

$$\langle h_0 \rangle_{N+1} = \frac{\langle h_0 \cosh \beta h_0 \rangle_N}{\langle \cosh \beta h_0 \rangle_N}$$

We have relationship between N+1 system and N system...

We can also now use a R.S. ansatz..

Notice we can write $J_{0j} = \frac{J_0}{N} + \delta J_{ij}$ (Subtract mean)

$$\langle h_0 \rangle_N = \sum_j J_{0j} \langle S_j \rangle_N$$

$$\langle (\delta h_0)^2 \rangle_N = \sum_{j,k} \delta J_{0j} \delta J_{0k} \langle \delta S_j \delta S_k \rangle$$

where $\delta S_i \equiv S_i - \langle S_i \rangle$

It is useful to now write

$$J_{ij} = \frac{J_0}{N} + \delta J_{ij}$$

$$\langle h_0 \rangle_N = J_0 \frac{1}{N} \sum \langle S_j \rangle_N + \sum_j \delta J_{ij} \langle S_j \rangle_N$$

If we now "average" over disorder we we that

$$[\langle h_0 \rangle_N] = J_0 m_N \qquad m_N = \frac{1}{N} \sum_{j=1}^{N} \langle S_j \rangle_N$$

$$[\langle (\delta h_0)^2 \rangle_N] = \sum_{j,k=1}^{N} [\delta J_{0j} \delta J_{0k}] \langle \delta S_i \delta S_j \rangle_N$$

$$= \frac{J^2}{N} \sum_{j=1}^{N} \langle (\delta S_j)^2 \rangle_N$$

Self averaging $\Rightarrow \langle S_i \rangle_N^2 = \frac{1}{N} \ldots$

$$q_N = \frac{1}{N} \sum_{i=1}^{N} \langle S_i \rangle \ldots$$

$$= \frac{J^2}{N} \sum_{j=1}^{N} [\langle (S_i - \langle S_i \rangle)^2 \rangle] = J^2 [1 - q_N]$$

$$= J^2 [1 - q] \text{ (large N)}$$

This tells us that

$$p^N(h_0) = \frac{1}{\sqrt{2\pi J(1-q)}} \cdot e^{-\left[\frac{(h-\langle h_0\rangle_N)^2}{2J(1-q)}\right]}$$

So now we can substitite this into our self-consistency condition..

$$\langle h_0\rangle_{N+1} = \frac{\langle h_0 \cos\beta h_0\rangle_N}{\langle \cosh \beta h_0\rangle_N}$$

$$\langle S_0\rangle_{N+1} = \frac{\langle \sinh \beta h_0\rangle_N}{\langle \cosh \beta h_0\rangle_N}$$

$\int$

$$\langle e^{\beta h_0}\rangle_{Gaussian} = e^{\beta\langle h_0\rangle_N + \beta^2 J(1-q)}$$

$$\langle \sinh \beta h_0\rangle_N = e^{\frac{\beta^2 J(1-q)}{2}}\, 2\sinh\beta\langle h_0\rangle_N$$

$$\langle \cosh \beta h_0\rangle_N = e^{\frac{\beta^2 J(1-q)}{2}}\, 2\cosh\beta\langle h_0\rangle_N$$

$$\langle S_0\rangle_{N+1} = \tanh[\beta\langle h_0\rangle_N] \quad \left(\text{Large N limit the same}\right)$$

Furthermore, $\frac{\partial}{\partial\beta}\langle \sinh\beta h_0\rangle = \langle h_0 \cosh \beta h_0\rangle$

$$= \beta J(1-q)\, e^{\frac{\beta^2 J(1-q)}{2}}\, 2\sinh\beta\langle h_0\rangle_N$$

$$+ \langle h_0\rangle_N\, e^{\frac{\beta^2 J(1-q)}{2}}\, 2\cosh\beta\langle h_0\rangle_N$$

$$\langle h_0\rangle_{N+1} = \langle h_0\rangle_N + \beta J(1-q)\tanh \beta\langle h_0\rangle_N$$
$$= \langle h_0\rangle_N + \beta J(1-q)\langle S_0\rangle_{N+1}$$

Subtituting

$$\langle S_0\rangle_{N+1} = \tanh\left[\beta\left(\langle h_0\rangle_{N+1} - \beta J^2(1-q)\langle S_0\rangle_{N+1}\right]\right.$$

Self averaging ≈ self consistent equations..

$$\langle S\rangle = \tanh\left[\beta\langle h_0\rangle - \beta J^2(1-q)\langle S\rangle\right]$$

B.P... actually use

$$\langle h_0\rangle = \sum_j \{\delta J_{0j} S_j\}$$ and "not average over disorder"

$$\boxed{\langle S_0\rangle = \tanh\left[\beta J_0 m + \sum_j \delta J_{0j}\langle S_j\rangle - \beta J^2(1-q)\langle S_0\rangle\right]}$$

This will be starting point of our BP equations...