$\boxed{\text{Technical Tools: Dynamical Mean Field Theory and Chaos in Generalized Lotka Volterra Equations}}$

In this lecture, we will ask what happens beyond the region where we have Replica Symmetry Breaking (RSB)

Recall, last time we analyzed the fixed points of Generalized Lotka Volterra

$$\dot{n} = \frac{r n_i}{K_i}\left(K_i - n_i - \sum_{j\neq i} \hat{A}_{ij} n_j\right) + r n^{min}$$

For analysis today will need to add small "immigration rate"

$\Rightarrow$ Had it last time too when we said

$$N_0 = \max\left[0, \frac{r_i^{eff}}{1-\rho\sigma^2}\right]$$

We actually rescale... T

$$N_i \equiv \frac{n_i}{K_i} \qquad \frac{\hat{A}_{ij} \cancel{K_j}}{K_i} \equiv \alpha_{ij}$$

$$\lambda \equiv \frac{n_0^{min}}{K}$$ (assume to be small and same for all species

$t \to rt$

Get.. (taking r_i and K_i to be same for all species)

$$\dot{N}_i = N_i\left(1 - N_i - \sum_{j\neq i} \alpha_{ij} N_j\right) + \lambda$$

This model has some amazing phenomenology...

~~Beyond~~ As before we study this model for Random α_{ij}.

but for the special case where the enteries are uncorrelated

$$\langle \alpha_{ij}\alpha_{kl}\rangle - \langle \alpha_{ij}\rangle\langle\alpha_{kl}\rangle = \sigma^2\left[\frac{\delta_{ik}\delta_{jl}}{S} + \rho\frac{\delta_{il}\delta_{jk}}{S}\right]$$

$$\langle \alpha_{ij}\rangle = \frac{\mu}{S}$$

$\boxed{\text{Take } \rho = 0}$

Basic phenemonology is same when $\rho \neq 0$ but as we will see this allows us to write a DMFT for this model..

So what is basic phenomenology...

Recall we derived the R.S. breaking criteria last lecture

$$\frac{1}{\sigma^2} = \frac{(1+\rho)}{\sqrt{2}} \xrightarrow{\rho=0} \frac{1}{\sigma^2} = \frac{1}{\sqrt{2}}$$

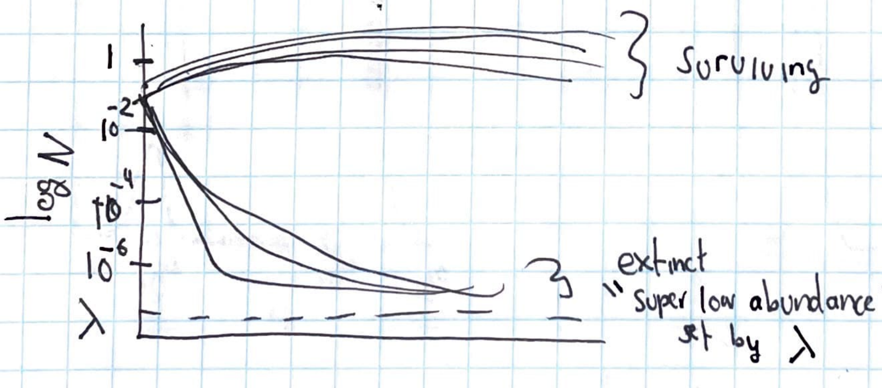
Or exactly when the $\mathrm{Var}(\alpha) = \frac{\sigma^2}{S}$

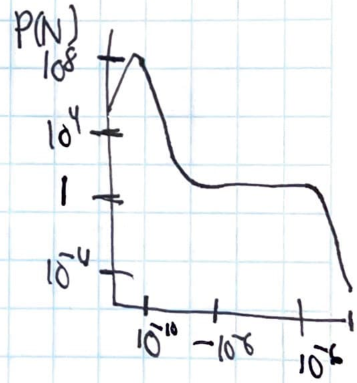
$$\left(S\,\mathrm{Var}(\alpha) < \sqrt{2} \right)$$

This is precisely May Criteria for stability

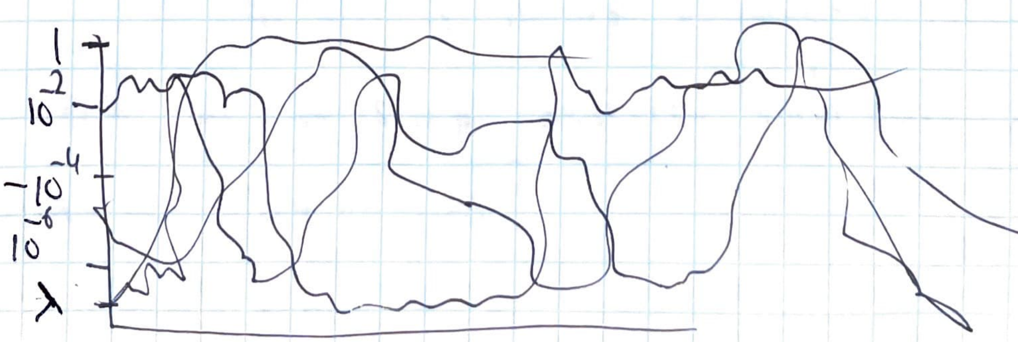
So we see that number of surviving species S^* have S^* less than May bound

R.S.



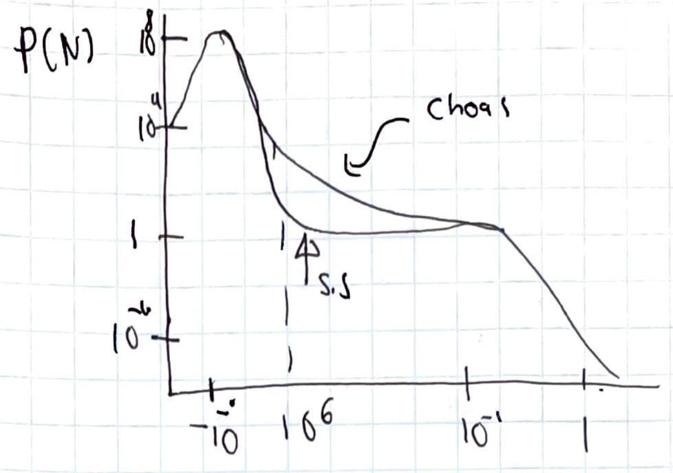


Beyond RS Transition Chaos with "boom + bust"



(low abundance boom to high abundance and then go back down

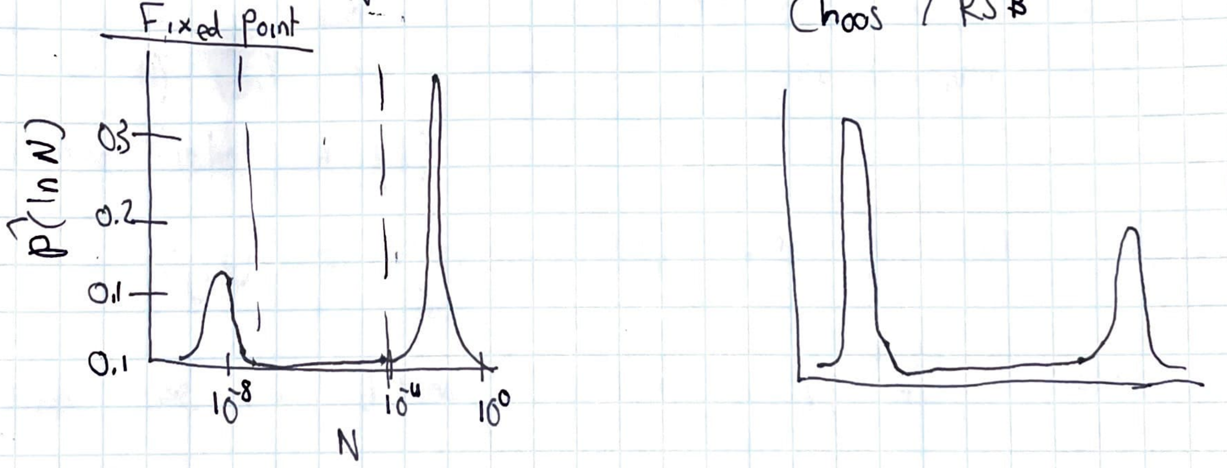
In this chaotic phase $P(N)$ actually looks surprisingly just like steady-state



Actually useflu to define probability density for $\log N$

$$\hat{P}(\log N)\, d(\log N) = P(N)\, dN$$

$$\frac{\hat{P}(\log N)}{N} = P(N)$$



Both are peaked... suggests that $\ln N$ is natural variable...

How can we possibly describe such complicated chaotic dynamics?...

The basic idea of DMFT is to do a "dynamic" version of the cavity...

Let us introduce a new species N_0 and ask how the dynamics of remaining system changes...

Again, we can do this perturbatively

Introduce new species N_0

$$\frac{dN_i}{dt} = N_i\left(r_i^{(t)} - N_i - \sum_{j \neq i} \alpha_{ij} N_j - \alpha_{i0} N_0\right) + \lambda$$

(we have introduced r_i which we will set all to zero...)

$$\frac{dN_0}{dt} = N_0\left(r_0 - N_0 - \sum_j \alpha_{0j} N_j\right) + \lambda$$

$$= N_0\left(r_0 - N_0 - m(t) - \sum a_{0j} N_j\right) + \lambda$$

$m(t) = \frac{1}{S}\sum_j N_j(t)$

As before we can view $\alpha_{i0} N_0(t)$ as a small perturbation of $N_i(t)$ but now viewed as a time-dependent perturbation on the field $r_i(t)$

$$r_i(t) \rightarrow r_i(t) + \delta r_i(t) \Rightarrow \delta r_i(t) = -\alpha_{i0} N(t)$$

Now we can define two-time susceptibility matrix

$$\left[\nu_{jk}(t,t') = \frac{\delta N_j(t)}{\delta r_k(t')}\right]$$

Clearly

$$\nu_{jk}(t,t') = 0 \quad \text{if} \quad t < t'$$

For $t > t'$, we assume that this is a "Gaussian Process" $\Rightarrow$ Essentially two-time functions can be described as one big Gaussian...

From this we see that as before we can write

$$N_j(t) = N_{j\backslash 0}(t) + \sum_k \int dt'\, \nu_{jk}(t,t')\, a_{k0}\, N_k(t')$$

Substituting this in original equation we have that

$$\frac{dN_0}{dt} = N_0\Big(1 - N_0 - m(t) - \sum_j a_{0j} N_{j\backslash 0}(t) + \sum_{jk} a_{0j} a_{k0} \int dt'\, \nu_{jk}(t,t')\, N_k(t')\, N_j(t)$$

Now as before we will average. Since this is a double sum..

In general very complicated. If

$$\langle a_{0j}\, a_{k0}\rangle = \rho \frac{\sigma^2}{S} \delta_{jk}$$

(ρ is correlation
$\rho = 1$ Symmetric
$\rho = -1$ Anti-symmetric
$\rho = 0$ Uncorrelated

$$\underbrace{\rho \frac{\sigma^2}{S}}_{=0} \sum_j \int dt'\, \nu_{jj}(t,t')\, N_j(t)\, N_j(t')$$

However, for $\rho = 0$... This TAP correction disappears and we can say something...)

In particular, in this case are equations become

$$\frac{dN_0}{dt} = N_0\Big(1 - N_0 - m(t) - \overbrace{\sum a_{0j} N_{j10}}^{}\Big)$$
$$\delta r_0(t)$$

But we know that

$$\langle \delta r_0(t) \rangle = \sum_j \langle a_{0j} \rangle N_{j10} = 0 ,$$

$$\langle \delta r_0(t)\, \delta r_0(t') \rangle = \sum_{jk} \langle a_{0j}\, a_{0k} \rangle N_{j10}(t)\, N_{k10}(t')$$
$$= \sigma^2 \langle N_{j10}(t)\, N_{j10}(t') \rangle$$

Where we have defined the empirical expectation value

$$\langle N_{j10}(t)\, N_{j10}(t') \rangle = \frac{1}{S} \sum_{j=1}^{S} N_{j10}(t)\, N_{j10}(t')$$

Thus, we are almost there, we can now invoke self averaging to get..

(A) $\langle N_{j10}(t)\, N_{j10}(t') \rangle = \langle N_0(t)\, N_0(t') \rangle$

(B) $m(t) = \langle N_0(t) \rangle$

To get a self-consistency equation.. (now dropping zero subscript)

$$\frac{dN}{dt} = N(t)\big[1 - N(t) - \mu m(t) + \sigma \xi(t)\big] + \lambda$$

where $m(t) = \langle N(t) \rangle$ and $\langle \xi(t)\, \xi(t') \rangle = \langle N(t) N(t') \rangle$

τ

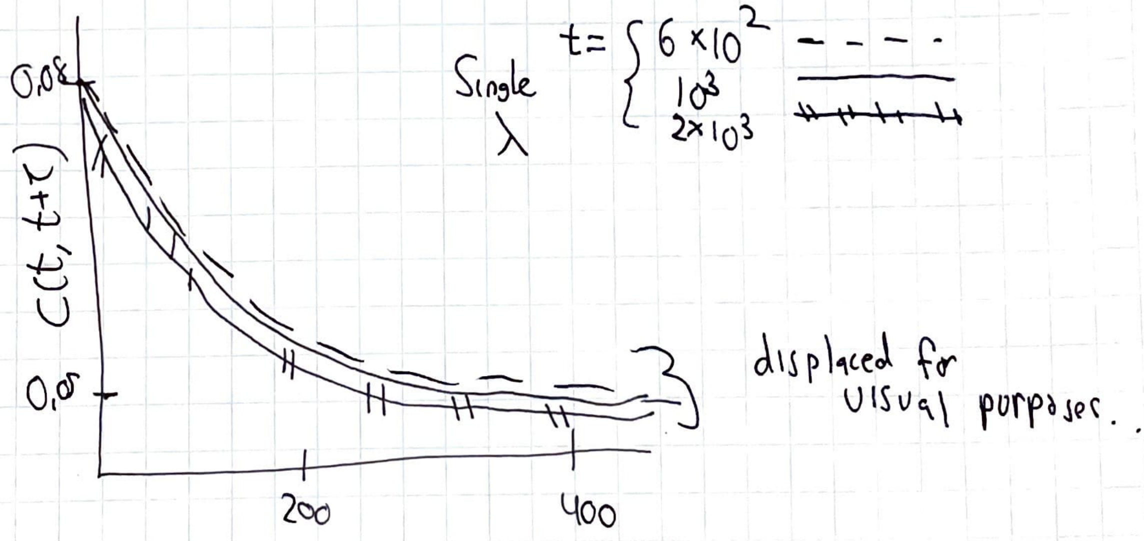
$\xi(t)$ is a Gaussian process...

(In other words it is defined by the first two moments $\langle \xi(t)\rangle$ $\langle \xi(t)\,\xi(t')\rangle$...

So how do we proceed? We need some ansatz for the correlation function

$$\langle N(t)\, N(t')\rangle \equiv C(t,t')$$

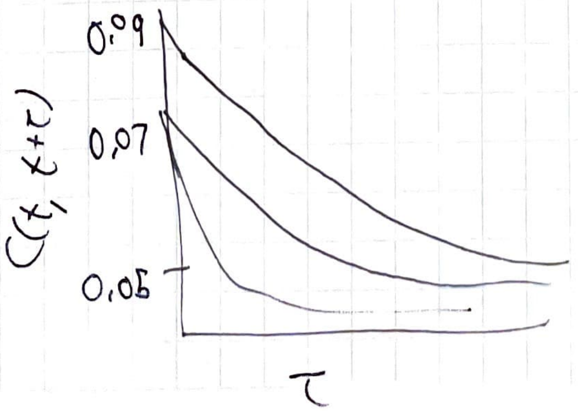
From numerics, we see that after sufficiently long time scale, the correlations no longer grow with t

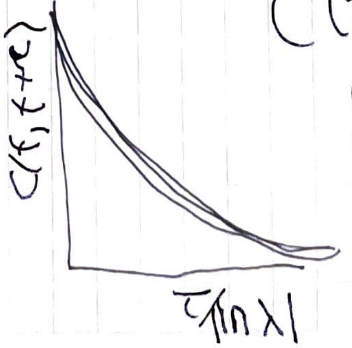


This implies

$$C(t,t') \approx C(t-t') \quad \text{for } t \text{ long enough...}$$

Actually key observation, is that for different λ, the $C(t, t+\tau)$ collapses if $\tau \to \frac{\tau}{\ln \lambda}$





What is $|\log \lambda| \Rightarrow$ time it takes species starting out at N^{min} to reach $O(1)$ abundance...

So we have $C_\lambda(t, t+|\ln\lambda| s) \rightarrow \hat{C}(s)$...

The second general observation is that we should work in rescaled abundance...

Recall, that the plots are all most natural in terms of the ~~rescaled~~ quantity $\ln N$... This motivates us to define

$$z = \frac{\ln N_i}{\ln \lambda}$$

(in non-rescaled variables this is just $\dfrac{\ln \frac{n_i}{K}}{\ln \frac{n^{min}}{K}}$)

$$s = \frac{t}{|\ln \lambda|}$$

very strange but understandable... Log ratio of min and actual abundance.

In terms of these rescaled variables we have that we can rewrite our DMFT as

$$\frac{1}{N(t)}\frac{dN}{dt} = \overbrace{[1 - m(t) + \sigma\xi(t)] - N(t)}^{g(t)} + \frac{\lambda}{N(t)}$$

(effective growth rate at low abundances)

self interaction

$$z'(s) = g(s) + \exp\{-|\ln\lambda|(z(s)+1)\} - \exp\{|\ln\lambda| z(s)\}$$

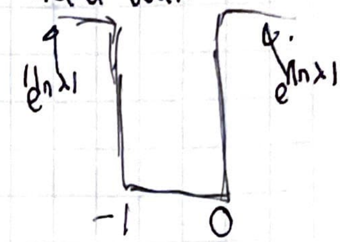
as we take $\lambda \to 0^+$

$$\lim_{\lambda\to 0^+} e^{-|\ln\lambda|(z+1)} = \begin{cases} 0 & \text{if } z > -1 \\ +\infty & \text{if } z < -1 \end{cases}$$

$$\lim_{\lambda\to 0^+} e^{|\ln\lambda| z} = \begin{cases} 0 & \text{if } z < 0 \\ +\infty & \text{if } z > 0 \end{cases}$$

$\approx g(s) \approx N(s)$

So we see that as $\lambda \to 0^+$, this becomes a hard wall and we can see



random walker

$$z'(s) = g(s) + W(z-1) - W(z)$$

where
$$W(z) = \begin{cases} \infty & z > 0 \\ 0 & z < 0 \end{cases}$$

We further know that

$$\langle g(s)\, g(s')\rangle - \langle g(s')\rangle\langle g(s)\rangle = \sigma^2 \langle N(s) N(s')\rangle$$
$$= \sigma^2 \underbrace{C(s-s')}_{\text{Translationally invariant..}}$$

(want to get self-consistent equation for $g(s) \Rightarrow$ not $N(s)$ on right..

When $z < 0$ it is clear $N(s) = \exp[|\ln \lambda|\, z(s)] = 0$

But what do we do in limit $z \to 0$ and $\lambda \to 0^+$

We need to impose boundary conditions at $z = 0$..

Here we know that

[since $z'(s)$ does not $= \infty$..] $z'(s) = 0$ when $z = 0$ (Infinite barrier..)

So $g(s) = W(z)) \Rightarrow$ $N(s) = g(s)$

So that $\underbrace{N(s) = g(s)\,\theta(z(s))}_{\text{Impeneterable...}}$ with $\theta(0) = 1$..

Same as saying high abundance fixed points are in approximate steady-state $\dot{N} = N(g(s) - N(s)) \lambda$

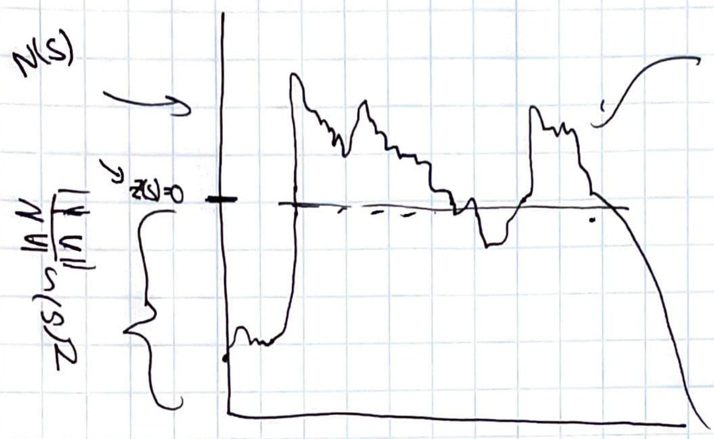
$\Rightarrow g(s) \approx N(s)$

$$\langle g(s)\rangle = 1-\mu\langle g(s)\,\theta(z(s))\rangle$$

$$\langle g(s)g(s')\rangle - \langle g(s')\rangle\langle g(s)\rangle = \sigma^2\langle g(s)g(s')\,\theta(z(s))\,\theta(z(s'))\rangle$$

Self consistency equations for $g(s)$

So what does this look like... Brown



$g(s)$ is Brownian when $z(s)>0$

For $z<0$ $z'(s) = g(s)$ smooth

For $z>0$ $N(s) \approx g(s)$ (rough)

$N(t) \approx 0$ for high abundance species... once $z(s) > 0$ we have rough Brownian motion..

Our approximation is that we have ignored immigration

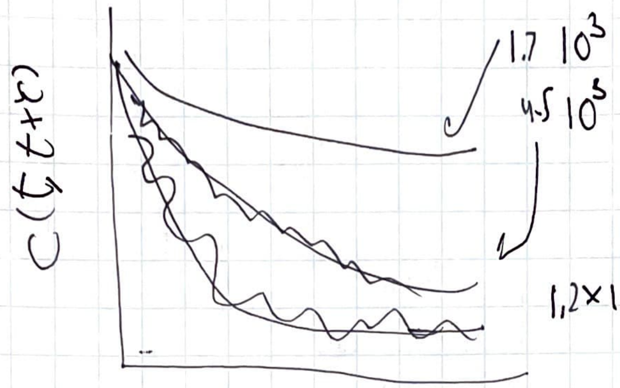
Slowly low abundance species rise to high abundance destabilize high abundance species (competition) and the cause crash..

Boom + Bust dynamics..

Something qualitatively different happens when
$\lambda = 0$ (strictly no immigration)

System exhibits aging...

Data Collapse



Collapse with $\frac{\tau}{t}$ ... $C_{\lambda=0}(t, t+tt') \to \hat{C}(t')$

In other words the system exhibits <u>aging</u>

<u>Linear growth of correlation time with age of the system</u>

This is common phenomena in glasses..

So we have seen GLVs are a nice place to learn
- $\Rightarrow$ Cavity
- $\Rightarrow$ Susceptibility related to resolvent RMT
- $\Rightarrow$ RS breaking when solution becomes unstable
- $\Rightarrow$ A soluable example of DMFT...