The Generalized Lotka Volterra Equation
R.S. breaking + Cavity

Last lecture we introduced the G.L.V. equations

ρ is "non-reciprocity"

$$\frac{dN_i}{dt} = \frac{g_i N_i}{r_i}\left(r_i - N_i - \sum A_{ij} N_j\right) \qquad i = 1 \dots S$$

$$r_i \sim N(\bar{r}, \sigma_R^2) \qquad A_{ij} = \frac{\mu}{S} + \sigma a_{ij} \qquad \langle a_{ij} \rangle = 0 \qquad \langle a_{ij} a_{kl} \rangle = \frac{1}{S}\left(\delta_{ik}\delta_{jl} + \rho\, \delta_{il}\delta_{jk}\right)$$

The key to these equations is that species could go extinct. i.e $N_i = 0$ was fixed point.

Hence, D.O.F's can "appear" and "disappear".

We has S pecies initially in region species pool and some $S^* \leq S$ can survive...

Analyzing this is trying to understand the relationship between "diversity" (how big S^*) is and stability (is the ecosystem stable to small perturbations...)

There are conceptually <u>two different kinds</u> of perturbations...

(1) Say the system reaches a fixed point $\vec{N}^*$. We can $\vec{N}^* \to \vec{N}^* + \delta\vec{N}$ and ask if this grows or shrinks (usual stability for dynamical system / ODE)

(2) A second kind of stability is perturbation to changes in parameters.. we will be concerned primarily with

$$\vec{r} \to \vec{r} + \delta\vec{r}$$

These are related and we will come back to this..

Since May's argument, lots of questions about how mathematically we see large diverse systems since for "generic linerized dynamics"

$$\frac{d\vec{N}}{dt} = (I - L)\delta\vec{N} \quad i = 1, \ldots S^*$$

Linearized matrix assummed random

we expect $L_{ij} \sim N(0, \sigma_L^2)$

there to be transidion for S^* sufficiently big! (positive eigenvalue)..

To answer this question, we can ask about steady states of G.L.V.

Random $r_i \sim N(\bar{r}, \sigma_r^2)$

$$0 = N_i \left(r_i - N_i - \sum_j A_{ij} N_j\right)$$

We can define an effective carrying capacity r_i^{eff}

$$r_i^{eff} = r_i - \sum_j A_{ij} N_j = r_i - \mu\langle N\rangle - \sum_j a_{ij} N_j$$

Notice that this clearly a "Mean-field" since it is sum over many variables..

Also notice that this is "almost linear" problem we solved...

We will need one more criteria... We require the steady-state to be non-invasible.. Namely if there exists $N_i^* > 0$, we take that steady state... $0 = N_i (r_i^{eff} - N_i)$

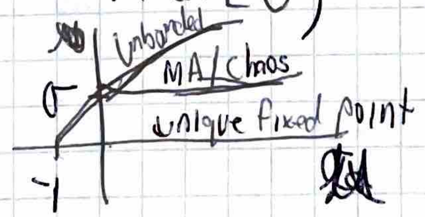
So that

$$N_i^* = \max\Big[0,\ \underbrace{r_i - N_i - \sum A_{ij} N_j}_{\text{Solution of linear equations}}\Big]$$

$$N_i^* = \max\left[0,\ r^{eff}_{i(\vec{N})}\right]$$

just $r_i^{eff} < 0$ means species goes extinct..

Phase Diagram



So now let us as before assume R.S. ...
this is just the statement that

r_i^{eff} should be modeled as Gaussian with mean and variance...

As before to calculate this mean and variance we imagine "invading" the ecossystem with new Species N_0 . . .

$$0 = N_i(r_i - N_i - \sum_{j\neq i} A_{ij} N_j - A_{i0} N_0) = N_i (r_i - N_i - \mu\langle N\rangle - \sum a_{ij} N_j - a_{i0} N_0 - \frac{\mu}{S} N_0)$$

$$0 = N_0 (r_0 - N_0 - \sum_{j=1}^{N} A_{0j} N_j)$$

We proceed almost identical to as before...

Notice $r_i \rightarrow r_i + \delta r_i$ with $\delta r_i = -A_{i0} N_0$

So we can define susceptibility matrix..

$$\nu_{ij} = \frac{\partial N_i}{\partial r_j}$$

and write

$$N_i = N_{i\backslash 0} - \sum_{j=1}^{S} \nu_{ij} a_{j0} N_0$$

Sustituting we get to leading order in $1/S$

$$0 = N_0 [r_0 - N_0 - \mu\langle N\rangle - \sigma \sum a_{0j} (N_{j\backslash 0} - \sigma \sum_k \nu_{jk} a_{k0} N_0)]$$

There are two solutions for N_0

$$N_0 = 0 \qquad N_0 = \frac{\bar{r} + \delta r_0 - \sigma \sum a_{0j} N_{j\backslash 0}}{1 - \sigma^2 \sum_{ij} \nu_{ij} a_{0j} a_{i0}}$$

Since we are taking large S limit we can substitute

$$\sum_{j,l=1}^{S} a_{0j} a_{l0} v_{lj} = \sum_{j,l}^{S} \langle a_{0j} a_{l0} \rangle v_{lj}$$

$$= \frac{\rho}{S} \sum_{j} v_{jj} = \rho \tilde{v}$$

$$\tilde{v} = \frac{1}{S} \sum_{j} v_{jj} \;\ldots$$

$$N_0 = \max\left(0, \frac{\overbrace{r + \delta r_0 - \sigma \sum_{i,j} a_{0j} N_{j\backslash 0} - \mu \langle N \rangle}^{\text{Gaussian Field R.S. } r_0^{eff}}}{1 - \sigma^2 \rho \tilde{v}}\right)$$

$$\langle r_0^{eff} \rangle = r - \mu \langle N \rangle \;\ldots \qquad \langle N \rangle = \frac{1}{S} \sum_j N_j$$

$$\langle \delta r_0^{eff} \rangle = \sigma_R^2 + \sigma_A^2 \langle N^2 \rangle \qquad \langle N^2 \rangle = \frac{1}{S} \sum_j N_j^2$$

$$N_0 = \max\left(0, \frac{r - \mu \langle N \rangle + \sqrt{\sigma_R^2 + \sigma_A^2 \langle N^2 \rangle}\, \overset{\text{Unit normal variable}}{Z_N}}{1 - \rho \sigma^2 v}\right)$$

As before we will invoke self-averaging and see species distribution is self-consistent truncated Gaussian

Let us call fraction of surviving species $\frac{S^*}{S} = \phi_N$

then we can

$$\cancel{\phi_{N_0}} = \phi_N = \frac{1}{S}\sum_{i=1}^{S} \Theta(N_{i,0})$$

$$\langle N_0 \rangle = \langle N \rangle = \frac{1}{S}\sum_{i=1}^{S} N_{i,0}$$

$$\langle N_0^2 \rangle = \langle N \rangle = \frac{1}{S}\sum_{i=1}^{S} N_{i,0}^2$$

$$\left\langle \frac{dN_0}{dr_0} \right\rangle = \frac{1}{S}\sum_i v_{ii}$$

So notice

$$\tilde{v} = \left\langle \frac{\partial N_0}{\partial r_0} \right\rangle = \frac{\phi_N}{1-\rho\sigma^2 v}$$

Non-zero fraction since

$$\frac{\partial N_0}{\partial r_0} = 0 \quad \text{if } N_0 = 0$$

$$\frac{\partial N_0}{\partial r_0} = \frac{1}{1-\rho\sigma^2 v} \quad \text{if } N_0 \neq 0$$

Useful to define $y = \max\left(0, \frac{c}{b}z + \frac{a}{b}\right)$

$$\langle y^j \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\frac{a}{c}}^{\infty} e^{-\frac{x^2}{2}} \left(\frac{c}{b}x + \frac{a}{b}\right)^j dx$$

$$= \left(\frac{c}{b}\right)^j \frac{1}{\sqrt{2\pi}} \int_{-\frac{a}{c}}^{\infty} e^{-\frac{x^2}{2}} \left(x + \frac{a}{c}\right)^j dx$$

$$= \left(\frac{c}{b}\right)^j W_j\left(\frac{a}{c}\right)$$

where $$W_j(\Delta) = \frac{1}{2\pi}\int_{-\Delta}^{\infty} e^{-\frac{z^2}{2}}(z+\Delta)\,dz$$

(Truncated Gaussian distribution integral)

Need identity

In terms of $W_j(\Delta)$ get $\quad W_2(\Delta) = W_0(\Delta) + \Delta$

$$v = \frac{\phi_N}{1-\rho\sigma^2} \qquad \phi_N = W_0\left(\frac{r-\mu\langle N\rangle}{\sqrt{\sigma_r^2+\sigma^2\langle N^2\rangle}}\right)$$

$$\langle N\rangle = \frac{\sqrt{\sigma_r^2+\sigma^2\langle N^2\rangle}}{1-\rho\sigma^2 v} W_1\left(\frac{r-\mu\langle N\rangle}{\sqrt{\sigma_r^2+\sigma^2\langle N^2\rangle}}\right)$$

$$\langle N^2\rangle = \frac{\sigma_R^2+\sigma^2\langle N^2\rangle}{(1-\rho\sigma^2 v)^2} W_2\left(\frac{r-\mu\langle N\rangle}{\sqrt{\sigma_r^2+\sigma^2\langle N^2\rangle}}\right)$$

This allows us to calculate M.F. R.S. distribution exactly,

Truncated Gaussian with means, moments, and non-zero fraction given by above self-consistent equations..

So far we have not said anything about the Stability of this Replica Symmetric ansatz..

We would like the M.F. Solution to exist and be stable to perturbations..

Happens numerically as σ_c increased

If it is not stable, then we say that there is Replica-Symmetry breaking...

$\Longrightarrow$ check when infinitely susceptible to small perturb.

To do this we imagine taking the surviving species and perturbing them

strength

$$\vec{N}^* \longrightarrow N^* + \epsilon \eta$$

random vector

$\langle \vec{\eta} \cdot \vec{\eta} \rangle \sim$ finite $\bar{\eta}^2$

We can calculate the effect this will have by looking at our cavity species

$$N_0 = \max \left[0, \ \frac{r_0 - \mu \langle N \rangle - \sigma \sum a_{0j} N_{j\setminus 0} + \epsilon \eta_0}{1 - \rho \sigma^2 \nu} \right]$$

We would like to measure sensitivity to perturbation

$$\left\langle \left(\frac{dN_0}{d\epsilon} \right)^2 \right\rangle$$

(square because want to know when variance) diverges

Notice that

$$\frac{dN_0}{d\varepsilon} = 0 \qquad \text{if } N_0 = 0$$

$$\frac{dN_0}{d\varepsilon} = \frac{-\sigma \sum_{j=1}^{S^*} a_{0j} \frac{\partial N_{j\backslash 0}}{\partial \varepsilon} + \sum \eta_{j} a_{0j}}{1 - p\sigma^2 v}$$

$\bar{\eta}^2 = \sum_j \eta_{j,n}^2$

$$\left\langle \left(\frac{dN_0}{d\varepsilon}\right)^2 \right\rangle_{a+} = \frac{\sigma^2 \frac{S^*}{S} \frac{1}{S^*} \sum_{j=1}^{S^*} \left(\frac{\partial N_{j\backslash 0}}{\partial \varepsilon}\right)^2 + \bar{\eta}_0^2 \sigma_\varepsilon^2}{(1 - p\sigma^2 v)^2}$$

(Over non zero) value

which we choose to scale to be finite

Variance of random variable

We now write

$$\frac{1}{S^*} \sum_{j=1}^{S} \left(\frac{\partial N_{j\backslash 0}}{\partial \varepsilon}\right)^2 = \left\langle \left(\frac{\partial N_{j\backslash 0}}{\partial \varepsilon}\right)^2 \right\rangle_+$$

Non-zero average

But self averaging say

$$\left\langle \left(\frac{\partial N_{j\backslash 0}}{\partial \varepsilon}\right)^2 \right\rangle_+ = \left\langle \left(\frac{\partial N_0}{\partial \varepsilon}\right)^2 \right\rangle_+$$

So that

$$\left[(1 - p\sigma^2 v)^2 - \sigma^2 \phi_s\right] \left\langle \left(\frac{\partial N_0}{\partial \varepsilon}\right)^2 \right\rangle_+ = \bar{\eta}_0^2 \frac{\sigma^2}{(1 - p\sigma^2 v)}$$

$$\Rightarrow \left\langle \left(\frac{\partial N_0}{\partial \varepsilon}\right)^2 \right\rangle_+ = \frac{\bar{\eta}_0^2 \, \sigma^2 / (1 - p\sigma^2 v)}{\left[(1 - p\sigma^2 v)^2 - \sigma^2 \phi_s\right]}$$

This susceptibility diverges when

$$(1-\rho\sigma_c^2 v_c)^2 = \sigma_c^2 \phi_s$$

(c - critical subscript) v_c, σ_c

Use

$$\phi_s = v_c(1-\rho\sigma^2 v)$$

To get

$$\underbrace{(1-\rho\sigma^2 v_c)}_{\text{Non-zero since by assumption R.S. solution exists}}\underbrace{(1-\rho\sigma_c^2 v_c-\sigma_c^2 v_c)}_{\to 0} = 0$$

zero

$$\boxed{v_c = \frac{1}{\sigma_c^2(1+\rho)}}$$

We still need another equation to solve for v_c to do this we note

$$(1-\rho\sigma_c^2 v_c) = \left(1-\frac{\rho\sigma_c^2}{\rho\sigma_c^2+\sigma_c^2}\right) = \frac{\sigma_c^2}{\rho\sigma_c^2+\sigma_c^2} = \frac{1}{1+\rho}$$

~~From this we find that~~

(HW) and use identity $w_2(\Delta)=w_0(\Delta)+\Delta$ to find

$$\boxed{\frac{1}{\sigma_c^2} = \frac{1+\rho}{\sqrt{2}}}$$