Technical Note II: Zero Temp Cavity + Relationship to RMT.

We want to solve for $(\lambda I - A)\vec{x} = b$ in limit $N \to \infty$
$A = A^T$

So last time we derived the self-consistency equation

$$X_0 = \frac{b_0 - \sum_j A_{0j} X_{j\backslash 0}}{\lambda - \sigma_A^2 \nu}, \quad \nu_{00} = \frac{\partial X_0}{\partial b_0} = \frac{1}{\lambda - \sigma_A^2 \nu}$$

Recall we had matrix susceptibility

$$\nu_{ij} = \frac{1}{N} \sum \left[\frac{\partial X_i}{\partial b_j} \right]$$ Change in solutions if we change $\vec{b}$

We also defined an average trace

$$\nu = \frac{1}{N} \sum \nu_{jj}$$

The numerator is just a random variable (for different draws of A_{0j}, b_0)

whose means is $\bar{b} + \mu \langle X \rangle$ where

$\langle x \rangle = \frac{1}{N} \sum x_j$ since $\langle A_{0j} \rangle = \frac{\mu}{N}$ and $X_{j\backslash 0}$ is independent of A_{0j}

It's variance is just $\mathrm{Var}(b_0) + \mathrm{Var}\left(\sum A_{0j} X_{j\backslash 0}\right)$ since these are idependent random variables

$\mathrm{Var}(b_0) = \sigma_b^2$

$$\mathrm{Var}\left(\sum_{j=1}^{N} A_{0j} X_{j\backslash 0}\right) = \sum_{k=1}^{N} \sum_{j=1}^{N} \langle A_{0j} A_{0k} \rangle X_{j\backslash 0} X_{k\backslash 0} = \sum_{j,k} \frac{\delta_{jk} \sigma_A^2}{N} X_{j\backslash 0} X_{k\backslash 0}$$

So that

$$\mathrm{Var}\left(\sum_{j=1}^{N} A_{0j} X_{j\backslash 0}\right) = \sigma_A^2 \langle X^2 \rangle \quad \text{where} \quad \langle X^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} X_j^2$$

So we can write X_0 as a random variable of the form

$$X_0 = \frac{\bar{b} + \mu \langle X \rangle + \sqrt{\sigma_B^2 + \sigma_N^2 \langle X^2 \rangle}\, Z}{\lambda - \sigma_A^2 \nu}$$

So now we invoke <u>self-averaging</u> ... For large system, assume that averaging over parameters $\{A_{ij}, b_i\}$ (different realizations)

Is the same as averaging over different X_i ...

Namely,

$$[X_0] = \langle X \rangle \qquad \langle X^2 \rangle - \langle X \rangle^2 = \frac{\sigma_B^2 + \sigma_N^2 \langle X^2 \rangle}{(\lambda - \sigma_A^2 \nu)^2}$$

$$[X_0^2] = \langle X^2 \rangle \qquad \langle X \rangle = \frac{\bar{b} + \mu \langle X \rangle}{\lambda - \sigma_A^2 \nu}$$

$$[\nu_{00}] = \nu \implies \nu = \frac{1}{\lambda - \sigma_A \nu}$$

We can solve these

$$\langle X^2 \rangle - \langle X \rangle^2 = \nu^2 (\sigma_B^2 + \sigma_N^2) \langle X^2 \rangle$$

$$\langle X \rangle = (\bar{b} + \mu \langle X \rangle) \nu$$

$$\lambda \nu - \sigma_A \nu^2 = 1 \implies \sigma_A^2 \nu - \lambda \nu + 1 = 0$$

$$\nu = \frac{\lambda \pm \sqrt{\lambda^2 - 4\sigma_A^2}}{2\sigma_A^2} \implies$$ Must be real so $|\lambda| > 2\sigma_A$

So we can see

$$\langle x \rangle = \frac{\bar{b} v}{1-\mu v}$$

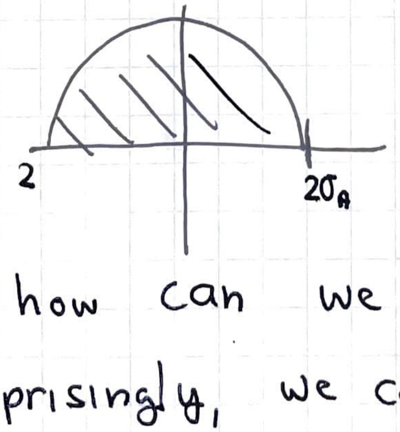
$$\langle x^2 \rangle = \frac{\langle x \rangle^2}{1 - v^2(\sigma_B^2 + \sigma_A^2)} = \frac{\bar{b}^2 v^2}{(1-\mu v)^2 [1 - v^2(\sigma_B^2 + \sigma_A^2)]}$$

$$v = \frac{\lambda \pm \sqrt{\lambda^2 - 4\sigma_A^2}}{2\sigma^2}$$

So we have self-consistently solved for the mean and variance....

So why does $|\lambda| > 2\sigma_A$... Recall, to have a good solution that we cannot have zero eigenvalues..

Actually, in a second we will see that symmetric Random matrix A with $A_{ij} \sim N(0, \frac{\sigma_A^2}{N})$ has spectrum described by Wigner Semi-Circle Law



So how can we calculate this spectrum...

Surprisingly, we can also use the calculation we just did to calculate the spectrum of A..

How? The key realization are probably familiar to you if you have ever taken a many-body course or a Field theory course...

Let us go back to a very simple identity

$$\frac{1}{x+i\epsilon} = \frac{1}{x+i\epsilon}\cdot\frac{x-i\epsilon}{x-i\epsilon} = \frac{x}{x^2+\epsilon^2} - i\frac{\epsilon}{x^2+\epsilon^2}$$

Notice that in the limit $\epsilon \to 0$ that

$$\lim_{\epsilon\to 0}\frac{\epsilon}{x^2+\epsilon^2} = -\pi\,\delta(x)$$

So that we have

$$\operatorname{Im}\frac{1}{x+i\epsilon} = -\pi\,\delta(x)$$

So what does this have to do with spectrum of matrices..

Notice that

$$\lim_{N\to\infty}\frac{1}{N}\operatorname{Tr}\operatorname{Im}\left[\frac{1}{x+i\epsilon+A}\right]$$

... we can work in diagnonal basis...

matrix

$$\approx -\pi\,\rho(\lambda_A)$$

Density of Spectrum of a matrix...

In other words

$$G_A(\lambda) = \frac{1}{\lambda I - A}$$

("Propagator")

$$\frac{1}{\pi}\operatorname{Im}\frac{1}{N}\operatorname{Tr}G_A(x-i\epsilon) = \rho_A(x)$$

] Spectrum...

$$\rho_A(x) = \lim_{N\to\infty} -\frac{1}{\pi}\operatorname{Im}\frac{1}{N}\left[\operatorname{Tr}\frac{1}{x+i\epsilon+A}\right]$$

But notice that if we return to our equations

$$(\lambda I + A)\vec{x} = \vec{b}$$

$$\lambda x_i + \sum_j A_{ij} x_j = b_i$$

Susceptibility

And differentiate w.r.t b_k we have

$$\lambda \frac{\partial x_i}{\partial b_k} + \sum_j A_{ij} \frac{\partial x_j}{\partial b_k} = \delta_{ik} \quad \Rightarrow \quad \nu_{jk} = \partial$$

In matrix form

$$(\lambda I + A)\nu = I$$

$$\nu(\lambda) = (\lambda I + A)^{-1}$$

~~So we we see that~~ $\cancel{-\nu(-\lambda) = G_A(\lambda)}$

So we see that

$$\lim_{\epsilon \to 0} -\lim_{N\to\infty} \frac{1}{\pi}\frac{1}{N} \operatorname{Im} \operatorname{Tr} \nu(x + i\epsilon) = \rho_A(\lambda)$$

But $\frac{1}{N}\operatorname{Tr}\nu = \frac{1}{N}\sum_j \nu_{jj}$] We calculated this quantity in the cavity equation..

$$\cancel{\frac{1}{2\pi} \pm \sqrt{x^2 - 4\sigma_A^2}}$$

Look at $\nu(x + i\epsilon) = \dfrac{x \pm \sqrt{x^2 - 4\sigma_A^2}}{2\sigma_A^2}$

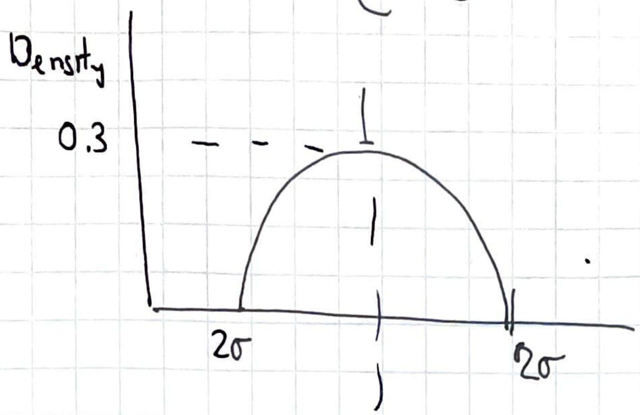
Need imaginary part... Only exists if

$|x| \leq 2\sigma_A$ $\quad \operatorname{Im}\nu \neq 0$

$|x| > 2\sigma_A$ $\quad \operatorname{Im}\nu = 0$

So we get that

$$\rho_A(x) = \begin{cases} \frac{1}{2\pi\sigma_A^2}\sqrt{4\sigma_A^2 - x^2} & \text{if } x \leq 2\sigma \\ 0 & \text{if } x \geq 2\sigma \end{cases}$$



"Wigner Semi-circle Law" !!

So our cavity susceptibility actually is closely related to the Resolvent $\frac{1}{z-A}$ whose trace is related to density...

This was a linear equation so nothing super complicated can happen...

But key was → Gaussian M.F.] Replica
→ Self-averaging] Symmetry...

So we have introduced two of the key tools of disordered M.F.T.
Cavity method + R.M.T.

We will return to these over and over again in increasingly complicated settings...

To introduce the idea of Replica Symmetry breaking and more complicated ideas, we have to actually move to a slightly more complicated model...
(with non-linearities) $\Longrightarrow$ R.S breaking, D.M.F.T.

The Generalized Lotka-Volterra Model. and Ecology..

$$\frac{dN_i}{dt} = \frac{r_i N_i}{K_i} \underbrace{\left(K_i - N_i - \sum_{j \neq i} A_{ij} N_j\right)}_{\text{Grow}} + \underbrace{\lambda_i}_{\text{small immigration rate } \lambda_i \to 0^+}$$

(Consider species abundances)

(regularize problem)

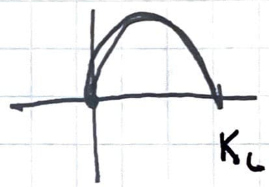
growth rate

K_i - Carrying Capacity

(A_{ij} measured in units of "A_{ii}" self-interactions) $A_{ij} \to$ species - specie interactions...

Look at when $A_{ij} = 0$... No other species ..

$$\frac{dN_i}{dt} = \frac{r_i N_i}{K_i} (K_i - \underset{\sim}{N_i})$$



Limitation due to Comptetition with self..

Two fixed point $N_i = 0$ (extinct) and $\underbrace{N_i = K_i}_{\text{at carrying capacity}}$

Main new feature.. <u>Species can go extinct</u>

Useful to consider somewhat simpler problem where r_i are all same and we set to 1...

$$0 = \frac{dN_i}{dt} = N_i \left(K_i - N_i - \sum_{j=1} A_{ij} N_j\right) \ldots$$

$$K = N_i + \sum_j A_{ij} N_j \,\Big]$$

Same equation we had but if this is negative "species disappear"

Essentially linear equations where D.O.F can disappear...

Lots of interest in understanding large, diverse ecosystem

Idea: Let draw K_i A_{ij} from some distribution..

$$K_i \sim N(\bar{K}, \sigma_K)$$
$$A_{ij} \sim N\left(\frac{\mu}{N}, \frac{\sigma_A}{\sqrt{N}}\right)$$

Same thermodynamic trick...

$$\Longrightarrow \langle A_{ij} A_{kl} \rangle = \delta_{ik}\delta_{jl}\frac{\sigma^2}{N} + \rho$$

Interested in knowing what happens for σ large, $\rho \neq 0$.. ect.

$$\frac{dN_i}{dt} = N_i\left(K_i - N_i - \sum_{j \neq i} A_{ij} N_j\right)$$

Define effective carrying capacity

$$K_i^{eff} = \underbrace{K_i - \sum_{j \neq i} A_{ij} N_j}$$

Notice this is "Mean Field"...

As before we will start with

R.S. solution

before moving on...

Notice when $\rho \neq 0$ A_{ij} no longer symmetric

$\downarrow$

this is no longer gradient

$$\frac{dN_i}{dt} = \nabla V(N)$$

"non-reciprocal" interactions + out of equilibrium...