Disordered Systems: Technical Tools I

We will start by introducing some technical tools
$\Rightarrow$ Namely how to do M.F.T. for disordered systems
(Cavity Method $\Rightarrow$ (zero temperature first))
And it's relationship to R.M.T.
We will then introduce another tool...
"Replicas"

And armed with that we will start analyzing some systems that are well described by M.F.T
$\rightarrow$ S.K. Model + Hopfield
$\rightarrow$ Generalize Lotka Voterra Equations

After that we will move beyond MFT, and talk about finite dimensional systems, .. focusing on Disorder RG + Scaling ..

Lets do some math:

We will be interested in solving following system of equations

$$(\lambda I + A)\vec{x} = b \quad \text{where} \quad A = A^T \ (\text{real E.V.})$$

and A is $N \times N$ matrix

$\vec{x}$ and $\vec{b}$ and N-dimensional vectors

Let us assume

$$\langle A_{ij} \rangle = \frac{\mu}{N}$$

$$\langle A_{ij} A_{kl} \rangle = (\delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}) \frac{\sigma_A^2}{N}$$

Intereseted in statistical propert[ies] as $N \to \infty$

And for now

$$b = \mathcal{N}(\bar{b}, \sigma_b^2)$$

Why this weird scaling with N ~~for~~ mean and variance?

Well we need a good limit as $N \to \infty$
(Good thermodynamic limit)

Notice in component form

$$\lambda x_i + \underbrace{\sum_{j=1}^{N} A_{ij} x_j} = b_j$$

Notice that this N terms so scales

$$\sim \quad N \langle A_{ij} x_j \rangle \sim$$

So inorde for this to have good limit (mean) need to scale $\frac{\mu}{N}$ . . .

But first moment could converge, also need second moment to be well defined.. Similar argument for variance is why we need second moment to scal as $1/N$

$$\sum_{jkl} A_{jk} x_k x_l A_{lj}$$

$$\sim \sum_{kl} x_k x_l \sum_j \langle A_{jk} A_{lj} \rangle$$

$$\sim \sum_k x_k^2 \frac{\sigma^2}{N} \sim \langle x^2 \rangle \sigma^2$$

What we would like to do is understand is the Solution to this

$$X = (\lambda I + A)^{-1} b$$

For example can we (1) say something about when these solutions exist? (2) Statistical Properties of $\{\vec{X}\} \to$?

~~for~~ (3) And relatedly, eigenvalues and eigenvectors of A?

The reason we need λ is that by making it large enough or small enough we can prevent $(\lambda I + A)^{-1}$ from having <u>zero eigenvalue</u>?

Namely, $\lambda + \lambda_i^A \geq 0$ for all i if A has "bounded spectrum" $\lambda^{min} \leq \lambda_i^A \leq \lambda^{max}$ with Prob 1 as $N \to \infty$ ...

So what do we expect?

$$A X_i + \sum_j A_{ij} X_j = b_i$$

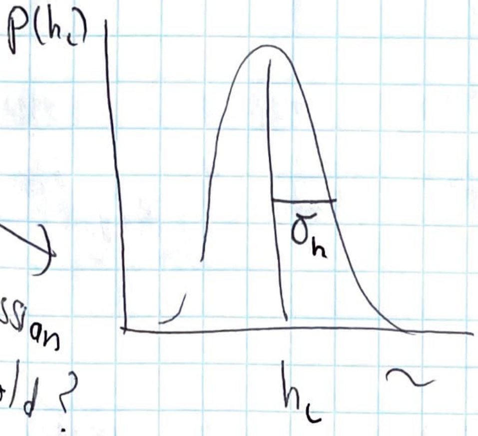
Notice that this is a mean-field Sum of many, many variables... what do we expect this to be a Gaussian

(Bold h_i for different i)

$h_i \sim$ 

We can do two kinds of plots:

Take a large system of size $M \times N$ $M, N \gg 1$
and make a plot of ~~the~~ the $M \times N$ h_i



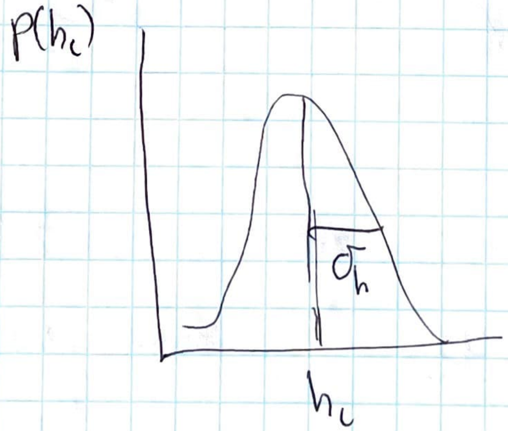
When wouldn't Gaussian assumption hold?

$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ w/ very different statistics..

(Gaussian Mean Field + Self Averaging = Replica Symmetric Ansatz)

$\sim$ has some mean $\bar{h}$ and std de σ_h

Or we can ~~calculate~~ ~~to~~ consider M copies of a system of size N and consider distribution of a $\underline{\text{single}}$ fixed $\underline{\underline{h_i}}$



Since all A_{ij} are drawn idependentally,

we expect the system to $\underline{\underline{\text{self-average}}}$

i.e. We expect these Gaussians to be identical...

We will make use of this...

So what are means and variance of this field?
And how could we use it to calculate statistics we want?

So naively (incorrect calculation)

$$\left\langle \sum_j A_{ij} x_j \right\rangle = \sum_j \langle A_{ij} x_j \rangle \overset{?}{=} \sum_{j=1}^{N} \langle A_{ij} \rangle x_j$$

$$\sim \underline{\mu} \langle x \rangle$$

Why is this wrong?

$$\langle x \rangle \sim \frac{1}{N} \sum_j x_j$$

Well actually $x_j(A_{ij}) \Rightarrow$ You can easily convince yourself (and we will see below)

Solution is a function of matrix elements

$$\langle A_{ij} x_j \rangle \sim \frac{\mu}{N} x_j + \mathcal{O}\left(\frac{1}{N}\right)$$

Correction of order $1/N$

But since we are summing this is an order 1 correction and cannot be ignored...

So how do we do this properly and take advantage of M.F.T. to both find statistics of solutions and spectrum of A..

Cavity Method: Idea due to Parisi, Mezard, Virasaro
later duelaped by Mezard, Zecchina, etc..

Basic idea relate a system with
size N to a system of size $N+1$
We expect the solutions to be <u>perturbatively</u>
related when $N \gg 1$...

So how does this work consider system with
new variable X_0

$$\left.\begin{aligned} &\lambda X_i + \sum_{j\neq i} A_{ij} X_j + \overbrace{A_{i0} X_0}^{\text{extra term}} = b_i \\ &\lambda X_0 + \sum_{j} A_{0j} X_j = b_0 \quad \Big] \text{ New equation..} \end{aligned}\right] \begin{matrix} N+1 \\ \text{System} \end{matrix}$$

Old Equations

$$\lambda X_{i0} + \sum_j A_{ij} X_{j0} = b_i \qquad i=1, \dots N$$

Notice that we can view the term $A_{i0}X_0$ as
order $\mathcal{O}(\frac{1}{N})$ perturbation.. $\delta b_i = -A_{i0} X_0$

So we can do perturbation theory? Ask how
does $\{X_i\}$ change when we change $\{\delta b_j\}$?

Notice we can write everything in terms of susceptibility matrices $\nu_{ij} = \frac{dX_i}{db_j}$ ...

$$\lambda \frac{dX_i}{db_k} + \sum_j A_{ij} \frac{dX_j}{db_k} = \delta_{ik}$$

In matrix form

$$(\lambda I + A)\nu = I \Rightarrow \nu = (\lambda I + A)^{-1}$$

So we see that susceptibility is related to inverse of our matrix...

So in particular, we can write relating solution for N+1 variables to solution for N variables as

$$\underbrace{X_j}_{\substack{\text{N+1 variable}\\ \text{solution}}} = \underbrace{X_{j\backslash 0}}_{\substack{\text{N-variable}\\ \text{solution}}} + \sum_k \nu_{jk}\,\delta b_k = X_{j\backslash 0} - \sum_k \nu_{jk} A_{k0} X_0$$

Why is this interesting

Notice $\langle A_{0j} X_{j\backslash 0}\rangle = \langle A_{0j}\rangle\langle X_{j\backslash 0}\rangle$

These are uncorrelated so we can solve our problem of the fact X_j depends of matrix elements.

Let us now write the extra equation as

$$\lambda X_0 + \sum_j A_{0j}\Big(X_{j\backslash 0} - \sum_k \nu_{jk} A_{k0} X_0\Big) = b_0$$

we can calculate moments of this easily

Which becomes

$$\Big(\lambda - \underbrace{\sum_{JK} A_{0J} A_{K0}}_{} v_{JK}\Big) X_0 = b_0 - \sum_J A_{0J} X_{J\setminus 0}$$

replace by expectation value to leading order in N

$$\Rightarrow \Big(\lambda - \sum_{JK} \frac{\delta_{JK}\,\sigma_A^2}{N} v_{JK}\Big) X_0 = b_0 - \sum_J A_{0J} X_{J\setminus 0}$$

$$\big(\lambda - \sigma_A^2 \bar{v}\big) X_0 = b_0 - \sum_J A_{0J} X_{J\setminus 0}$$

where we have defined $\bar{v} = \frac{1}{N}\sum_{J=1}^{N} v_{JJ}$ . . .

This yields the "self-consistency equation"

$$X_0 = \frac{b_0 - \sum_J A_{0J} X_{J\setminus 0}}{\lambda - \sigma_A^2 v}$$

We will talk about how to analyze this properly next class. . .