PY 541 Problem Set 2: HW Exercises Due on Sept. 29th

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Read he assigned material before class. Pre-class questions are due midnight the night before before Tuesday and Thursday classes and noon on Wednesdays when homeworks are not due (usually Thursdays), and otherwise should be turned in with the homeworks.

Welcome to PY 541 Statistical Physics. This is a graduate course that assumes you have taken an undergrad course on these topics (e.g. Thermal Physics, Statistical Physics, etc.). It also assume some basic familiarity with core mathematical areas: probability, linear algebra, vector calculus as well rudimentary programming skills (Python, Mathematica, etc). If you do not feel comfortable with these concepts, please come talk to me.

All exercises are from second edition of Jim Sethna's book available at:

`https://www.lassp.cornell.edu/sethna/StatMech/index.html`. There are also hints and code for computational exercises at this website.

Readings and Pre-class questions

Tuesday:
Read: Chapter 2
In-class question: 2.6 Fourier and Green
In-class question: 2.18 Absorbing boundary conditions
Wednesday:
Read: Chapter 3, Sec. 3.1 (Microcannonical) and 3.2 (Ideal Gas)
Pre-class question: 3.1 Temperature and energy
In-class: Typical Set Problem (see below)
Thursday:
Read: Chapter 3, Sec. 3.3-3.5
Pre-class question: 3.16 Taste, smell, and μ
In-class question: 3.2 Large and very large numbers
Lecture: Derivation of equilibrium relations.
Tuesday In-class question: 3.5 Hard sphere gas
In-class question: 3.11 Maxwell Relations

Typical Set problem

The fundamental postulate of statistical mechanics is that all microstates are equally likely. This seems like a very crazy idea – why should it work at all? I think the best argument for why statistical mechanics works is the idea of "typicality". As systems get complicated enough, all the states we are likely to encounter will be typical. Let us explore the idea using the simple idea of a coin toss.

Consider tossing a coin with probability p of getting heads N times. We will be interested in sequences of length N of the form $X = TTHHTT \dots TTH$.

(a) How many such sequences are there?

Now imagine we toss a coin and generate many, many of these sequence $\{X_1, X_2, \dots, X_M\}$ with $M \gg 1$. As N get very large we will see that only a very small fraction of these 2^N sequences will actually be seen in this set. We will call this subset of sequences that we do see much more often than others the typical set. The following steps explain why this is the case.

(b) By considering the mean and variance of the binomial distribution, argue that in the limit where N is large we can think of the expected number of heads N_H and tails you will see N_T as random variables of the form

$$N_H = pN + \sqrt{Np(1-p)}\eta \tag{1}$$
$$N_T = (1-p)N - \sqrt{Np(1-p)}\eta, \tag{2}$$

where η is a random normal variable (a random number draw from a Gaussian) with mean 0 and variance 1.

(c) Use these expressions to argue that in the limit $N \to \infty$, all sequences can be divided into two sets. A typical set of sequences with $N_H = Np$ heads which occur in the set $\{X_1, \dots, X_M\}$ with probability

$$P(X) = e^{-NH(p)}, \tag{3}$$

with

$$H(p) = -p \log p - (1-p) \log(1-p) \tag{4}$$

and a non-typical set with probability that is exponentially smaller that the typical set.

(d) Use the fact that all typical sequences are equally likely to conclude that there are $e^{NH(p)}$ such sequences all of which are equally likely.

(e) Show that the fraction of sequences that are in the set typical set f is

$$f = e^{-N(\ln 2 - H(p))}. \tag{5}$$

Plot $H(p)$ as a function of p and show that $H(p) \leq \ln 2$ with equality only for $p = 1/2$. What fraction of sequences are in typical set for $p = 0.1$? Thus, in most cases the typical set is a small fraction of the total number of sequences.
(f) What does this have to do with our postulates of Statistical Mechanics? What does this have to do with "compression" (e.g. zipping a file)?

For an interesting discussion of this in the context of Information theory and the idea of compression, see David Mackay Chapter 4.3-4.4 of his masterful book relating information theory to statistical physics https://www.inference.org.uk/mackay/itila/.

Homework Exercises

All hints are available at: https://www.lassp.cornell.edu/sethna/StatMech/EOPCHintsAndMaterials.html or ask me directly.

1. 2.5 Generating Random Walks.
2. 2.11 Stocks, volatility, and diversification.
3. 2.19 Run + tumble.
4. 2.20 Flocking
5. 3.19 Random energy model (Optional problem strongly encouraged for theorists)

Honor Code

All students are expected to follow the BU Honor Code. While collaboration is allowed and encouraged on HWs, each student should write up their own solutions. Copying HW is strictly forbidden. The students are allowed to consult all resources and books. However, students are NOT allowed to consult problem solutions from previous years or as found on the web.