Boundary-Value Problems in Cylindrical Coordinates

The solution of the Laplace equation in cylindrical coordinates is $\Phi = R(\rho)Q(\phi)Z(z)$, where the separate factors are given in the previous section. Consider now the specific boundary-value problem shown in Fig. 3.9. The cylinder has a radius a and a height L, the top and bottom surfaces being at $z = L$ and $z = 0$. The potential on the side and the bottom of the cylinder is zero, while the top has a potential $\Phi = V(\rho, \phi)$. We want to find the potential at any point inside the cylinder. In order that Φ be single valued and vanish at $z = 0$,

$$Q(\phi) = A \sin m\phi + B \cos m\phi$$
$$Z(z) = \sinh kz$$

where $\nu = m$ is an integer and k is a constant to be determined. The radial factor is

$$R(\rho) = CJ_m(k\rho) + DN_m(k\rho)$$

If the potential is finite at $\rho = 0$, $D = 0$. The requirement that the potential vanish at $\rho = a$ means that k can take on only those special values:

$$k_{mn} = \frac{x_{mn}}{a} \qquad (n = 1, 2, 3, \ldots)$$

where x_{mn} are the roots of $J_m(x_{mn}) = 0$.

Combining all these conditions, we find that the general form of the solution is

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \sinh(k_{mn}z)(A_{mn} \sin m\phi + B_{mn} \cos m\phi) \tag{3.105a}$$

At $z = L$, we are given the potential as $V(\rho, \phi)$. Therefore we have

$$V(\rho, \phi) = \sum_{m,n} \sinh(k_{mn}L)J_m(k_{mn}\rho)(A_{mn} \sin m\phi + B_{mn} \cos m\phi)$$

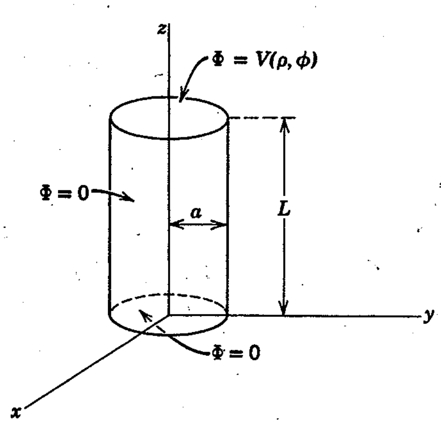

Figure 3.9

This is a Fourier series in ϕ and a Fourier–Bessel series in ρ. The coefficients are, from (2.37) and (3.97),

$$A_{mn} = \frac{2\,\text{cosech}(k_{mn}L)}{\pi a^2 J_{m+1}^2(k_{mn}a)} \int_0^{2\pi} d\phi \int_0^a d\rho\, \rho V(\rho, \phi) J_m(k_{mn}\rho) \sin m\phi$$

and (3.105b)

$$B_{mn} = \frac{2\,\text{cosech}(k_{mn}L)}{\pi a^2 J_{m+1}^2(k_{mn}a)} \int_0^{2\pi} d\phi \int_0^a d\rho\, \rho V(\rho, \phi) J_m(k_{mn}\rho) \cos m\phi$$

with the proviso that, for $m = 0$, we use $\frac{1}{2}B_{0n}$ in the series.

The particular form of expansion (3.105a) is dictated by the requirement that the potential vanish at $z = 0$ for arbitrary ρ and at $\rho = a$ for arbitrary z. For different boundary conditions the expansion would take a different form. An example where the potential is zero on the end faces and equal to $V(\phi, z)$ on the side surface is left as Problem 3.9 for the reader.

The Fourier–Bessel series (3.105) is appropriate for a finite interval in ρ, $0 \leq \rho \leq a$. If $a \to \infty$, the series goes over into an integral in a manner entirely analogous to the transition from a trigonometric Fourier series to a Fourier integral. Thus, for example, if the potential in charge-free space is finite for $z \geq 0$ and vanishes for $z \to \infty$, the general form of the solution for $z \geq 0$ must be

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \int_0^{\infty} dk\, e^{-kz} J_m(k\rho)[A_m(k) \sin m\phi + B_m(k) \cos m\phi] \tag{3.106}$$

If the potential is specified over the whole plane $z = 0$ to be $V(\rho, \phi)$ the coefficients are determined by

$$V(\rho, \phi) = \sum_{m=0}^{\infty} \int_0^{\infty} dk\, J_m(k\rho)[A_m(k) \sin m\phi + B_m(k) \cos m\phi]$$

The variation in ϕ is just a Fourier series. Consequently the coefficients $A_m(k)$ and $B_m(k)$ are separately specified by the integral relations:

$$\frac{1}{\pi} \int_0^{2\pi} V(\rho, \phi) \begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix} d\phi = \int_0^{\infty} J_m(k'\rho) \begin{Bmatrix} A_m(k') \\ B_m(k') \end{Bmatrix} dk' \tag{3.107}$$

These radial integral equations of the first kind can be easily solved, since they are *Hankel transforms*. For our purposes, the integral relation,

$$\int_0^{\infty} x J_m(kx) J_m(k'x)\, dx = \frac{1}{k}\, \delta(k' - k) \tag{3.108}$$

can be exploited to invert equations (3.107). Multiplying both sides by $\rho J_m(k\rho)$ and integrating over ρ, we find with the help of (3.108) that the coefficients are determined by integrals over the whole area of the plane $z = 0$:

$$\left.\begin{matrix} A_m(k) \\ B_m(k) \end{matrix}\right\} = \frac{k}{\pi} \int_0^{\infty} d\rho\, \rho \int_0^{2\pi} d\phi\, V(\rho, \phi) J_m(k\rho) \begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix} \tag{3.109}$$

As usual, for $m = 0$, we must use $\frac{1}{2}B_0(k)$ in series (3.106).

While on the subject of expansions in terms of Bessel functions, we observe that the functions $J_\nu(kx)$ for fixed ν, $\text{Re}(\nu) > -1$, form a complete, orthogonal (in k) set of functions on the interval, $0 < x < \infty$. For each m value (and fixed ϕ and z), the expansion in k in (3.106) is a special case of the expansion,

$$A(x) = \int_0^\infty \tilde{A}(k) J_\nu(kx)\, dk, \text{ where } \tilde{A}(k) = k \int_0^\infty x A(x) J_\nu(kx)\, dx \quad (3.110)$$

An important example of these expansions occurs in spherical coordinates, with spherical Bessel functions, $j_l(kr)$, $l = 0, 1, 2, \ldots$. For present purposes we merely note the definition,

$$j_l(z) = \sqrt{\frac{\pi}{2z}}\, J_{l+1/2}(z) \quad (3.111)$$

[Details of spherical Bessel functions may be found in Chapter 9.] The orthogonality relation (3.108) evidently becomes

$$\int_0^\infty r^2 j_l(kr) j_l(k'r)\, dr = \frac{\pi}{2k^2}\, \delta(k - k') \quad (3.112)$$

The completeness relation has the same form, with $r \to k$, $k \to r$, $k' \to r'$. The Fourier–spherical Bessel expansion for a given l is then

$$A(r) = \int_0^\infty \tilde{A}(k) j_l(kr)\, dk, \text{ where } \tilde{A}(k) = \frac{2k^2}{\pi} \int_0^\infty r^2 A(r) j_l(kr)\, dr \quad (3.113)$$

Such expansions are useful for current decay in conducting media or time-dependent magnetic diffusion for which angular symmetry reduces consideration to one or a few l values. See Problems 5.35 and 5.36.

3.9 *Expansion of Green Functions in Spherical Coordinates*

To handle problems involving distributions of charge as well as boundary values for the potential (i.e., solutions of the Poisson equation), it is necessary to determine the Green function $G(\mathbf{x}, \mathbf{x}')$ that satisfies the appropriate boundary conditions. Often these boundary conditions are specified on surfaces of some separable coordinate system (e.g., spherical or cylindrical boundaries). Then it is convenient to express the Green function as a series of products of the functions appropriate to the coordinates in question. We first illustrate the type of expansion involved by considering spherical coordinates.

For the case of no boundary surfaces, except at infinity, we already have the expansion of the Green function, namely (3.70):

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

Suppose that we wish to obtain a similar expansion for the Green function appropriate for the "exterior" problem with a spherical boundary at $r = a$. The result is readily found from the image form of the Green function (2.16). Using expansion (3.70) for both terms in (2.16), we obtain:

$$G(\mathbf{x}, \mathbf{x}') = 4\pi \sum_{l,m} \frac{1}{2l+1} \left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a} \left(\frac{a^2}{rr'} \right)^{l+1} \right] Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \quad (3.114)$$