Last week we used separation of variables to solve Helmholz equation in cylindrical coordinates. This week we will focus on spherical coordinates.

Consider Helmholz equation:

$$\nabla^2 \psi + k^2 \psi = 0.$$ 

In spherical coordinates

$$\nabla^2 \psi = \frac{1}{r^2 \sin \theta} \left[ \sin \theta \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right]$$

Since these are orthogonal coordinates, we know we can solve these equations using separation of variables.

$$\psi(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi)$$

Substituting and dividing by $\psi$

$$\frac{1}{R \rho^2} \frac{d}{dr} \left( \rho^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\Phi \sin^2 \theta} \frac{d^2 \Phi}{d\phi^2} = -k^2$$

Multiplying through by $\rho^2 \sin^2 \theta$ one has

$$\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = \rho^2 \sin^2 \theta \left[ -k^2 + \frac{1}{\rho^2 \sin \theta} \frac{d}{dr} \left( \rho^2 \frac{dR}{dr} \right) - \frac{1}{\rho^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) \right]$$

This depends only on $\phi$ and independent of $\theta$ and $r$.

Thus

$$\frac{d^2 \Phi}{d\phi^2} = -k^2 \Phi$$

The solution to this is

$$\Phi(\phi) = A \cos(k \phi) + B \sin(k \phi)$$

The boundary condition that $\Phi(\phi)$ be periodic requires

$$A = B$$

Thus

$$\Phi(\phi) = (A \cos(k \phi) + A \sin(k \phi))$$

This is a solution of the original Helmholtz equation with $R(r)$ and $\Theta(\theta)$ to be determined.
So we conclude:
\[
\frac{1}{\ell} \frac{d^2 \ell}{d\ell^2} = -m^2 \quad \text{for} \quad m \neq 0 \quad \{ e^{\text{im}\ell} \} \cup \{ \sin \text{im}\ell \} \cup \{ \cos \text{im}\ell \}
\]

And generally, \( m \) is integer.

And generally, \( m = 0 \) \( \rightarrow \) Constant \( \quad \text{(Generally)} \)

\[
\frac{1}{r^2 R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{r^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dR}{d\theta} \right) - \frac{m^2}{\ell \sin^2 \theta} = -\ell^2
\]

Multiplying through by \( r^2 \) and rearranging,

\[
Q = \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + r^2 k^2 = \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dR}{d\theta} \right) + \frac{m^2}{\sin^2 \theta},
\]

Depends only on \( r \) \quad Depends only on \( \theta \)

Gives rise to two more differential equations:

1. \[
\frac{1}{\sin \theta} \left( \sin \theta \frac{d}{d\theta} \sin \theta \frac{dR}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} + Q \sin \theta = 0
\]

2. \[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + k^2 R - Q R = 0
\]

These are the "associate Legendre Equation for \( Q = \ell(\ell + 1) \)" and the "spherical Bessel function" equation when \( k > 0 \). Can go from e to Q if

\[
l^2 = \ell(\ell + 1)
\]

bessel.pdf is justified by Strum-Liouville theory, but essentially stems from fact that any function fails if be expanded:

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{k^2 R - QR}{r^2}
\]
Consider first radial equation:

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + K^2 R - \frac{\ell(\ell+1)}{r^2} R = 0.
\]

Almost Bessel's equation but not quite.

Consider two cases:

\( k = 0 \) (Laplace's equation)

In this case we can plug in solution

\[ U = A r^{\ell+1} + B r^{-\ell} \]

Often seen in E.M.

For \( k \neq 0 \) case clearly pops up in Q.M.

\[ \nabla^2 \psi = \frac{\hbar^2 k^2}{2m} \psi \] Schrodinger Equation for particle in a sphere

In this case useful to substitute in

\[ R(kr) = \frac{Z(kr)}{(kr)^{\frac{\ell}{2}}} \]

\[
\frac{r^2 d^2 Z}{dr^2} + r \frac{dZ}{dr} + \left[ k^2 r^2 - (\ell + \frac{1}{2})^2 \right] Z = 0
\]

This is Bessel's equation with half integer \( \ell \).

So spherical-Bessel Functions

\[
J_\ell(x) = \sqrt{\frac{\pi}{2x}} J_{\ell + \frac{1}{2}}(x)
\]

\[
\tilde{J}_\ell(x) = \sqrt{\frac{\pi}{2x}} N_{\ell + \frac{1}{2}}(x)
\]

Spherical Bessel Oscillatory Functions

decaying oscillatory function
Once again we have orthogonality relations.

Recall
\[ \int_0^a J_\nu(xvn_\alpha^2) J_\nu(xvn_\alpha') dp = \frac{a^2}{2} J_{vn}^2(xvn) \delta_{nn'} \]

so that spherical Bessel's functions satisfy
\[ \int_0^a J_\nu(xvn_\alpha^2) J_{\nu'}(xvn_\alpha') dp = \frac{a^3}{2} [J_{\nu+1}(xvn)]^2 \delta_{nn'} \]

One can also show that
\[ \int_{-\infty}^{\infty} J_m(x) J_n(x) dx = 0 \quad n \neq n', \quad m/n \geq 0 \]

and
\[ \int_{-\infty}^{\infty} [J_n(x)]^2 dx = \frac{\pi}{2n+1} \]

We will use spherical Bessel functions in AW to do a QIM problem.

We now turn to the angular equation.
\[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) + \left[ e(\ell + 1) - \frac{m^2}{\sin^2 \theta} \right] \theta = 0 \]

Let us write \( x = \cos \theta \), \(-1 \leq x \leq 1\).
Then the equation becomes
\[ \frac{d}{dx} \left[ (1-x^2) \frac{dp}{dx} \right] + \left[ \ell(x+1) - \frac{\ell^2}{1-x^2} \right] p = 0. \]

This is called the associated Legendre Equation.

It is useful to first consider case \( m = 0 \). This arises when you have azimuthal symmetry.

Then this equation becomes Legendre equation
\[ \frac{d}{dx} \left[ (1-x^2) \frac{dp}{dx} \right] + \ell(x+1) p = 0. \]

Can solve this using series solution
\[ p(x) = x^\alpha \sum_{j=0}^\infty a_j x^j. \]

Substituting this in get
\[ \sum_j \left( (\alpha+j)(\alpha+j-1) a_j x^{\alpha-j-2} - (\alpha+j+1)(\alpha+j) a_j x^{\alpha-j-1} \right) = 0. \]

Look at terms proportional to \( \alpha - 2 \) and \( \alpha - 1 \)

\[ a_0 \alpha (\alpha - 1) = 0, \quad a_1 \alpha (\alpha + 1) = 0. \]

Notice two relations are equivalent

For general
\[ a_{j+2} = \left[ \frac{(\alpha+j)(\alpha+j+1) - \ell(x+1)}{(\alpha+j+1)(\alpha+j+2)} \right] a_j. \]

Consider \( \ell = 0 \)
\[ a_{j+2} = \frac{1}{j+2} a_j \quad \text{for} \quad a_0 \neq 0 \quad \text{then} \quad a_j \rightarrow \infty, \quad j \rightarrow \infty. \]
Can prove

Series converges for \( x^2 < 1 \) regardless of \( \ell \).
Series diverges at \( x = \pm 1 \) unless its finite
        and terminates.

The only way a series terminates is if \( \ell \in \mathbb{Z}^+ \) (positive integer)
        or 0

\[
\begin{cases} 
\ell \text{ even integer only} & \alpha = 0 \text{ series terminates} \\
\ell \text{ odd integer only} & \alpha = 1 \text{ series terminates}
\end{cases}
\]

So in both cases polynomials have \( x^\ell \) as their
        highest powers, next highest \( x^{\ell-2} \) down to \( x^0(x) \)
        for \( \ell \) even (odd).
        By convention choose \( q_0 \) so
        \( P_0(1) = 1 \).

This gives polynomials

\[
\begin{align*}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{3} (3x^2 - 1) \\
P_3(x) &= \frac{1}{2} [5x^3 - x] \\
P_4(x) &= \frac{1}{8} [35x^4 - 30x^2 + 3]
\end{align*}
\]

More generally can show series solution satisfies (See Arfken
        Chap 12)

\[
P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell
\]

(See graphs on next page)
Can also show
\[ \int_{-1}^{1} \, dx \, p_{e}'(x) p_{e}(x) = \frac{2}{2e+1} \delta_{e,e} \]

Furthermore, any function \(-1 \leq x \leq 1\) can be expanded in terms of them in a Legendre series representation

\[ f(x) = \sum_{e=0}^{\infty} A_e P_e(x) \]

\[ A_e = \frac{2e+1}{2} \int_{-1}^{1} f(x) p_e(x) \, dx \]

**Graphs**

- \( P_0(x) \)
- \( P_1(x) \)
- \( P_2(x) \)
- \( P_3(x) \)

\( P_e(x) \) has \( e \) zeros in the interval \(-1 \leq x \leq 1\).
Other useful recursion formulas

\[
\frac{dP_{l+1}}{dx} - \frac{dP_l}{dx} - (2l+1)P_l = 0
\]

\[
\Rightarrow (2l+1)P_{l+1} - (2l+1)XP_l + \ell P_{l-1} = 0
\]

\[
\Rightarrow \frac{dP_{l+1}}{dx} - x \frac{dP_l}{dx} - (l+1)P_l = 0
\]

\[
\Rightarrow (x^2 - 1) \frac{dP_l}{dx} - \ell x P_l + \ell P_{l-1} = 0
\]

Can use such formulas to do computations. WLOG \( \ell' = \ell \).

\[
I_1 = \int \frac{1}{x} x P_{\ell}'(x) P_{\ell}(x) dx
\]

\[
= \frac{1}{2l+1} \left[ \int_{-1}^{1} P_{\ell}'(x) [P_{\ell+1}(x) + \ell P_{\ell-1}(x)] dx \right]
\]

\[
= \begin{cases} 
\frac{2(l+1)}{(2l+1)(2l+3)} & \ell' = l+1 \\
\frac{2\ell}{(2\ell-1)(2\ell+1)} & \ell' = l-1
\end{cases}
\]

Such formulas are basis of selection rules in QNM.

\[
I_2 = \int x^2 P_{\ell}(x) P_{\ell}'(x) dx = \begin{cases} 
\frac{2(l+1)(l+2)}{(2l+1)(2l+3)(2l+5)} & \ell' = l+2 \\
\frac{2(2\ell^2 + 2\ell - 1)}{(2\ell-1)(2\ell+2)(2\ell+3)} & \ell' = \ell
\end{cases}
\]
Example

Consider

\[ f(x) = \begin{cases} +1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases} \]

\[ A_0 = \frac{2\ell + 1}{2} \int_0^1 P_\ell(x) \, dx - \int_{-1}^0 P_\ell(x) \, dx \]

Notice \( P_\ell(x) \) is odd for \( \ell \) odd and even for even \( \ell \). So only odd values contribute

\[ A_\ell = 0 \quad \ell \text{ even} \]

\[ A_\ell = (2\ell + 1) \int_0^1 P_\ell(x) \, dx \quad \ell \text{ odd} \]

To evaluate

\[ A_\ell = (2\ell + 1) \int_0^1 \frac{1}{2\ell !} \frac{d^{\ell+1}}{dx^{\ell+1}} (x^2 - 1)^\ell \, dx \]

Can do crazy recursion stuff using Rodriguez formula

\[ A_\ell = \frac{2\ell + 1}{2^\ell \ell !} \left[ \left. (x^2 - 1)^\ell \right|_0^1 \right] \int_0^1 f(x) = \frac{3}{2} P_\ell(x) \]

\[ -\frac{7}{8} P_{\ell-1}(x) + \frac{11}{16} \int_0^1 P_\ell(x) \]

In the modern world use Mathematica

\[ A_\ell = \frac{(-1)^{\ell+1}}{2^\ell (\ell+1)!} \frac{(2\ell+1) \, (2\ell+1)(\ell-2)!!}{2 \, (\ell+1)!} \]
Example

Consider two hemispheres at equal and opposite potentials

\[ V(\theta) = \begin{cases} V & 0 \leq \theta \leq \frac{\pi}{2} \\ -V & \frac{\pi}{2} < \theta \leq \pi \end{cases} \]

Find potential inside sphere

From separation of variables have \( k=0, \ m=0 \) so general solution is just

\[ \Phi(r, \theta) = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+1)}] P_l(\cos \theta) \]

Potential finite at \( r=0 \) so \( B_l = 0 \)

\[ \Phi(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) \]

We also have

\[ V(\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) \]

This is Legendre Series so

\[ A_l = \frac{2l+1}{2a^2} \int_{a}^{\infty} V(\theta) P_l(\cos \theta) \sin \theta \, d\theta \]

When \( V(\theta) = \begin{cases} V & (0 \leq \theta \leq \frac{\pi}{2}) \\ -V & (\frac{\pi}{2} < \theta \leq \pi) \end{cases} \)

coefficients proportional to example above so
\[ \Phi(r, \theta) = V \left[ \frac{3}{2a} P_1(\cos \theta) \right] - \frac{7}{8} \left( \frac{m^3}{a} \right) P_3(\cos \theta) + \frac{11}{16} \left( \frac{r}{a} \right)^7 P_5(\cos \theta) + \cdots \]

Potential outside sphere obtained by \( \frac{(r/a)^2}{(r/a)^2} \to \frac{(a/r)^2}{(a/r)^2} \).

Notice on \( z\)-axis \( P_0(\cos \theta) = (-1)^n \) so potential at \( \Phi(z=r) \) is related to general potential by multiplying coefficients \( A e^r \), \( B e^{-r} \) by \( P_0(\cos \theta) \). Explore this later to develop Green's Functions.

We now consider the more general case of \( m \neq 0 \).

Recall that using separation of variables, we arrived at the associated Legendre equation for \( x = \cos \theta \):

\[ \frac{1}{\sqrt{1-x^2}} \frac{d}{dx} \left[ (1-x^2) \frac{dP(x)}{dx} \right] + \left[ \ell(\ell+1) - \frac{m^2}{1-x^2} \right] P(x) = 0 \]

We saw that for \( m=0 \) the solutions were Legendre polynomials. Furthermore requiring \( P(x) \) to be finite at \( x=\pm 1 \) forced \( \ell \) to be a \( 0 \) or \( \mathbb{Z}^+ \).
Using some kind of arguments, it can be shown that in order for $P^m_l(x)$ to be finite on $-1 \leq x \leq 1$, $l = 0, 1, 2, \ldots$ and $m = -l, \ldots, l$.

In particular, one can show

$$P^m_l(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x)$$

$$= \frac{(-1)^m}{2^l l!} \left((1-x^2)^{m/2} \right) \frac{d^l}{dx^l} (x^2-1)^l$$

Furthermore, one can show

$$P^m_l(x) = (-1)^m P^m_l(x) \frac{(l-m)!}{(l+m)!} \frac{P^m_l(x)}{P_l(x)}$$

These functions are orthogonal

$$\int_{-1}^{1} P^m_l(x) P^m_l(x) = \frac{2 (l+m)!}{(2l+1)(l-m)!} \delta_{l'l}$$

More generally, we can consider $(\Theta, \phi)$ solution together to form spherical harmonics

$$Y^m_l(\Theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P^m_l(\cos \Theta) e^{im\phi}$$
These are the natural basis for spheres and have many nice properties:

\[ Y_{l,m}^*(\theta, \phi) = (-1)^m Y_{l,m}^*(\theta, \phi) \]

\[ \int d\theta \int d\phi \sin \theta Y_{l,m}^*(\theta, \phi) Y_{l,m}(\theta, \phi) = \delta_{l,l'} \delta_{m,m'} \]

One also has a completeness relation and expansion.

Any function

\[ g(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta, \phi) \]

\[ A_{lm} = \int d \cos \theta d\phi \ Y_{lm}^*(\theta, \phi) g(\theta, \phi) \]

Thus, a general solution to Laplace's equation is

\[ \phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ A_{lm} r^l + B_{lm} \frac{(-1)^l}{r^{l+1}} \right] Y_{lm}(\theta, \phi) \]

and to Helmholtz equation is

\[ E(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \kappa \left[ A_{lm} r^l e^{i\kappa r} + B_{lm} r^l e^{-i\kappa r} \right] Y_{lm}(\theta, \phi) \]
Let us get some intuition for spherical Harmonics and associated Legendre Polynomials, and Spherical Harmonics.

\[ l=0 \quad Y_{00} = \frac{1}{\sqrt{4\pi}} \]

\[ l=1 \]
\[ Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin \theta \cos \phi \]
\[ Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta \]

\[ l=2 \]
\[ \begin{align*}
Y_{22} &= \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta \cos^2 \phi \\
Y_{21} &= -\sqrt{\frac{15}{8\pi}} \sin^2 \theta \cos \phi \\
Y_{20} &= \frac{\sqrt{15}}{4\pi} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) 
\end{align*} \]

\[ l=5 \quad m=2 \]

\[ Y_{52} \]

\[ Y_{\text{lm}} \]

nodal lines →
= \( \text{m great circles} \)
through poles
and \( \text{m circles} \)
equal latitude

\[ + - - + \]
\[ + - - + \]
\[ + - + - \]
\[ + - + - \]
A very important result is the **Addition Theorem for spherical Harmonics**. We will use it to find expression for

\[ \frac{1}{|x-x'|} \]

in terms of spherical Harmonics.

In particular, one can show

\[ P_e(\cos \gamma) = \frac{4\pi}{(2\ell+1)} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta, \phi) \]

(See Arfken)

\[ P_e(\cos \gamma) = \sum_{\ell=0}^{\infty} (A_{\ell} r^{\ell} + B_{\ell} r^{-\ell+1}) P_{\ell}(\cos \gamma) \]

(Except at \( r=0 \))

So now we also know from Basic E.M.

\[ \Phi = \frac{q}{4\pi |x-x'|} = \sum_{\ell=0}^{\infty} (A_{\ell} r^{\ell} + B_{\ell} r^{-\ell+1}) P_{\ell}(\cos \gamma) \]

So now consider when \( \gamma = 0 \) and we are on \( z \)-axis.

If \( |x| > |x'| \) then we know \( \frac{1}{r^2} \) which has series expansion

\[ \frac{1}{r^2} = \sum_{\ell=0}^{\infty} \left( \frac{r'}{r} \right)^\ell \]

where \( P_{\ell}(r') \) is smaller (larger) of \( |x| \) and \( |x'| \).
Thus, we get coefficients $A_e = r^{l+1}$ and $B_e = 0$ when $r' > r$ and $A_e = 0$ and $B_e = r^l$ when $r > r'$. So that

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{l=0}^{\infty} \frac{r_l^2}{r_{l+1}^2} P_l(\cos \gamma)$$

Plugging in Addition theorem gives

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{r_{2l+1}} \frac{r_{2l+1}}{r_{4l+1}} Y_{lm}(\theta', \phi') Y_{lm}(\theta, \phi).$$

So we need to get an idea for how all this works. Let us do a detailed example.

Example

Consider two point charges $q$ and $-q$.

Let's find the electrostatic potential in spherical harmonics.

$$V = \frac{q}{4\pi \varepsilon_0} \left( \frac{1}{|\mathbf{x} - \mathbf{q}|} + \frac{1}{|\mathbf{x} + \mathbf{q}|} \right)$$

This problem has azimuthal symmetry so must independent of $\phi$.

$$= \frac{q}{\varepsilon_0} \sum_{l,m} \frac{1}{(2l+1)} \left( \frac{r_l^2}{r_{l+1}^2} \right) \left[ Y_{lm}(0, \phi) - Y_{lm}(\pi, \phi) \right] Y_{lm}(\theta, \phi)$$

hence only $m=0$ contributes.

$$= \frac{q}{\varepsilon} \sum_{l} \left( \frac{r_l^2}{r_{l+1}^2} \right) \left[ Y_{00}^x(0, \phi) - Y_{00}^x(\pi, \phi) \right] Y_{00}(\theta, \phi)$$

We take $|\mathbf{Q}| = \mathbf{P}_2$ fixed with $\mathbf{P}_2$ fixed.

This is a dipole moment.

\[ \begin{array}{c}
\rightarrow q \quad \Rightarrow \quad +q \\
\downarrow \\
\mathbf{P}_2 \quad \text{or} \quad -q \\
\end{array} \]
Since
\[ Y_{\ell}(\theta, \phi) = \frac{\sqrt{2\ell + 1} P_{\ell}(\cos \theta)}{\sqrt{4\pi}} \]
\[ Y_{\ell}^{\ell}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} P_{\ell}(\cos \theta) = (-1)^{\ell} \sqrt{\frac{2\ell + 1}{4\pi}} \]

So only odd \( \ell \) contribute

\[ = \frac{q}{2\pi \varepsilon_0} \sum_{\ell \text{ even}} \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell}(\theta, \phi) = \frac{q}{2\pi \varepsilon_0} \sum_{\ell \text{ even}} \frac{r_{\ell}^2}{2\ell + 1} P_{\ell}(\cos \theta) \]

Next let us take dipole limit. We are interested in

\[ \Phi = \frac{q \alpha}{2\pi \varepsilon_0} \sum_{k=0}^{\infty} \frac{(q/\alpha)^{2k+1}}{r^{2k+2}} P_{2k+1}(\cos \theta) \]

See every term except \( k=0 \) disappears

\[ \Phi = \frac{P}{4\pi \varepsilon_0} \frac{1}{r^2} P(\cos \theta) = \frac{P}{4\pi \varepsilon_0} \frac{P(\cos \theta)}{r^2} \]

This is the dipole moment (potential)

Look at Electric field
\[ \nabla = \frac{\partial}{\partial x} \hat{i} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{j} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \hat{k} \]

\[ + \frac{P}{4\pi \varepsilon_0} \left[ \frac{1}{r^3} \hat{r} P(\cos \theta) \hat{r} + \frac{\hat{\theta}}{r^2} \frac{\partial P(\cos \theta)}{\partial \theta} \right] \]
Example: Suppose now dipole moment is surrounded by grounded sphere. What is potential inside shell?

Well we can use the superposition principle to solve this problem.

Need solution & solution

\[ \nabla^2 \psi = 0 \quad \text{with} \quad \psi = 0 \quad \text{at} \quad r = a \]

General Solution

\[ \psi = \sum_{\ell} A_\ell r^\ell P_\ell (\cos \theta) \]

Now sum of two solutions is

\[ \frac{P_1 (\cos \theta)}{4\pi \varepsilon_0 r^2} + \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell (\cos \theta) \]

\[ \Psi (r=a) = 0 \]

So have

\[ \frac{P_1 (\cos \theta)}{4\pi \varepsilon_0 a^2} + \sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell (\cos \theta) = 0 \]

Since orthonormal apply \( l = 1 \) has non zero coefficient

\[ A_1 = \frac{-P}{4\pi \varepsilon_0 a^2} \quad m = \frac{P}{4\pi \varepsilon_0} \left( \frac{1}{r^2} - \frac{1}{a^2} \right) \cos \theta \]
You will do some hard-core B.V. problems for H.W.

A concise relationship between group theory and spherical harmonics can be found in

This concludes our overview of separation by variables.

**Integral Transforms**

For PDE's often useful to use integral transforms:

**Fourier transform**

**Laplace transform** → **Diffusion**

Let \( f(t) \) defined for \( t \geq 0 \)

\[
F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) \, dt
\]

\( S = \sigma + i\omega \)

Think of **Inverse - Laplace transform**

\[
f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i} \lim_{\gamma \to \infty} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) \, ds
\]

Properties:

**Linearity**

\[
 tf(t) \rightarrow -F'(s)
\]

\[
 t^n f(t) \rightarrow (-1)^n F^{(n)}(s)
\]

\[
 \frac{f(t)}{t} \rightarrow \int_0^\infty F(t) \, dt
\]

f(t) \rightarrow e^{as} F(s)

**Time shift**

\[
f(t-a) \rightarrow F(s) e^{as}
\]
Basis of Generating Functionals in probability,