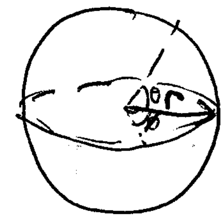
Lost week we used seperation of variables to solve Hemhotz equation is cylinderical coordinates. This week we will focus on Spherical coordinates.

Consider Hemholtz equation:



$$\nabla^2\psi + k^2\psi = 0.$$

In spherical coordinates

$$\nabla^2\psi = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right]$$

Since these spherical coordinates are orthognal coordinates, we know we can solve these equations using seperation of variables.

$$\psi(r,\theta,\varphi) = R(r)\,\Theta(\theta)\,\Phi(\varphi)$$

Susbtituting and dividing by ψ

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi r^2\sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = -k^2$$

Multiplying through by $r^2\sin^2\theta$ one has

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = \underbrace{r^2\sin^2\theta\left[-k^2 + \frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right]}_{\text{independent of } \varphi}$$

This depends only φ

2

Sign choice $\underline{\underline{\text{fixed}}}$ by periodicity

So we conclude:

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -m^2 \qquad m \neq 0 \quad \left\{ e^{\pm i m\phi} \right\} \text{ or } \left\{ \begin{matrix} \sin m\phi \\ \cos m\phi \end{matrix} \right\}$$

And generally m is integer

and

(Has azimuthal Symmetry) $m = 0 \longrightarrow$ Constant (Generally) $A + B\varphi$

$$\frac{1}{r^2 R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{r^2 \sin\theta\,\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{r^2\sin^2\theta} = -k^2$$

Multiplying through by r^2 and rearranging

$$Q = \underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + r^2k^2}_{\text{Depends only on } r} = \underbrace{-\frac{1}{\sin\theta\,\Theta}\frac{d}{d\theta}\left(\sin\frac{d\Theta}{d\theta}\right) + \frac{m^2}{\sin^2\theta}}_{\text{Depends only on } \theta} = Q$$

Gives rise to two more differential equations

- $$\frac{1}{\sin\theta}\left(\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right) - \frac{m^2\Theta}{\sin^2\theta} + Q\Theta = 0$$
- $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2R - \frac{QR}{r^2}$$

These are the "associate Legendre Equation for $Q = \ell(\ell+1)$ and the "spherical Bessel fuction" equation when $k>0$.

Can go from ℓ to Q

$$\ell = \frac{-1 \pm \sqrt{1+4Q}}{2}$$

~~The choice of $\ell(\ell+1)$ is justified by Strum-Louville theory but essentially stems from fact that any function $f(\theta)$ can be expanded in Legendre Spherical Harmonics~~

Consider first radial equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+k^2R-\frac{l(l+1)R}{r^2}=0.$$

almost Bessel's equation but not quite

Consider case l is an integer which we will see has to be true due to θ equation.

Consider two cases:

$k=0$ (LaPlace's equation)

In this case we can plug in solution

$$U = Ar^{l+1}+Br^{-l}$$

Often seen in E.M.

The $k\neq0$ case clearly pops up in Q.M.

$$\nabla^2\psi=\frac{\hbar k^2}{2m}\psi$$

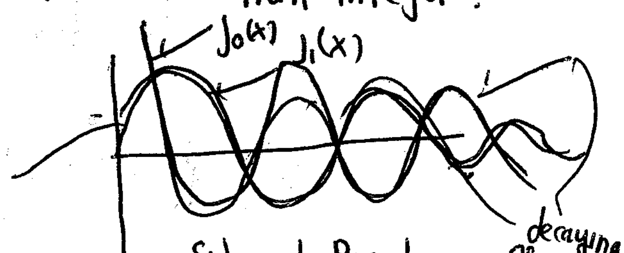
Schrondinger Equation for particle in a sphere

In this case useful to substitute in

$$R(kr)=\frac{Z(kr)}{(kr)^{1/2}}$$

$$r^2\frac{d^2Z}{dr^2}+r\frac{dZ}{dr}+\left[k^2r^2-\left(l+\tfrac{1}{2}\right)^2\right]Z=0$$

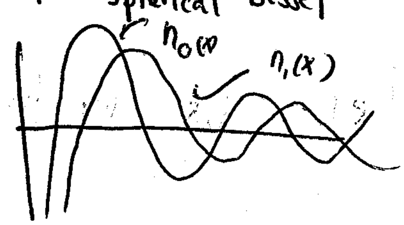
This is Bessel's equation with half integer!!



So spherical-Bessel Functions

$$j_n(x)=\sqrt{\frac{\pi}{2x}}\,J_{n+\frac{1}{2}}(x)$$

$$n_n(x)=\sqrt{\frac{\pi}{2x}}\,N_{n+\frac{1}{2}}(x)$$



Once again we have orthognality relations

Recall

$$\int_0^a J_\nu(X_{\nu n}\frac{\rho}{a}) J(X_{\nu n'}\frac{\rho}{a})\rho\, d\rho = \frac{a^2}{2} J_{\nu+1}^2(X_{\nu n})\,\delta_{nn'}$$

So that spherical Bessel's functions satisfy

$$\int_0^a d\rho\, \rho^2\, j_s(X_{sn}\frac{\rho}{a}) j_{s'}(X_{sn'}\frac{\rho}{a}) = \frac{a^3}{2}[j_{n+1}(X_{sn})]^2\delta_{nn'}$$

~~We can also expand a function in a ser~~

One can also show that

$$\int_{-\infty}^{\infty} j_m(x) j_n(x)\, dx = 0 \qquad m \neq n \quad m,n \geq 0$$

and $$\int_{-\infty}^{\infty} [j_n(x)]^2 dx = \frac{\pi}{2n+1}$$

We will use spherical Bessel functions in HW to do a Q.M. problem

We now turn to the angular equation.

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$$

Let us write $x = \cos\theta \qquad -1 \leq x \leq 1.$

Then the equation becomes

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0.$$

This is called the associated Legendre Equation.

It is useful to first consider case $m=0$. This arrises when you have azimuthal symmetry.

Then this equation becomes Legendre equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)P = 0$$

Can solve this using series solution

$$P(x) = x^{\alpha}\sum_{j=0}^{\infty} a_j x^j$$

Substituting this in get

$$\sum_j (\alpha+j)(\alpha+j-1)a_j x^{\alpha-j-2} - (\alpha+j+1)(\alpha+j)a_j x^{\alpha-j} - \ell(\ell+1)a_j x^{\alpha+j} = 0$$

Look at terms proportional to $\alpha-2$ and $\alpha-1$

$$\left.\begin{aligned} a_0\alpha(\alpha-1) &= 0 \\ a_1\alpha(\alpha+1) &= 0. \end{aligned}\right\}$$ notice two relations are equivalent

$a_1 \neq 0 \Rightarrow$ rewrite sum starting at $j=0$, sending $\alpha \to \alpha-1$

So take $a_1 = 0$, $a_0 \neq 0$, $\{\alpha = 0$ or 1.

For general j

$$a_{j+2} = \left[\frac{(\alpha+j)(\alpha+j+1) - \ell(\ell+1)}{(\alpha+j+1)(\alpha+j+2)}\right]a_j$$

Consider $\ell=0$ $\quad a_{j+2} = \frac{j}{j+2}a_j$ for a_0 then

$= a_0(1 + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ ← diverges

Can prove

Series converges for $x^2 < 1$ regardless of ℓ.

Series diverges at $x = \pm 1$ unless its finite and terminates.

Imposed by requiring well defined solutions at poles

does not diverge

The only way series terminates is is if $\ell \in \mathbb{Z}^+$ (positive integer or 0)

$$\begin{cases} \ell \text{ even integer} & \text{only } \alpha = 0 \text{ series terminates} \\ \ell \text{ odd integer} & \text{only } \alpha = 1 \text{ series terminates} \end{cases}$$

So in both cases polynomials have x^ℓ as their highest powers, next highest $x^{\ell-2}$ down to $x^0(x)$ for ℓ even (odd). By convention chose a_0 so $P_\ell(1) = 1$.

This gives polynomials

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{3}(3x^2 - 1)$$ (Dipole moment)
$$P_3(x) = \frac{1}{2}[5x^3 - x]$$
$$P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$$ (quadrapole moment)

More generally can show series solution satisfies (See Arfken Chap 12)

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2 - 1)^\ell$$

(SEE GRAPHS ON NEXT PAGE)

Can also show

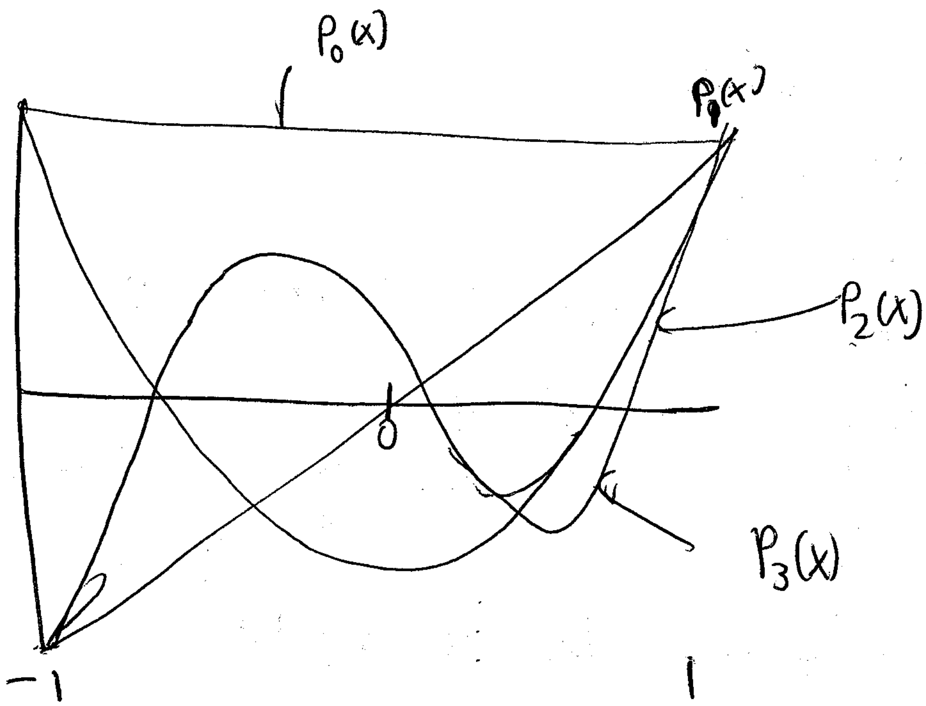
$$\int_{-1}^{1} dx\, P_{\ell'}(x) P_\ell(x) dx = \frac{2}{2\ell+1} \delta_{\ell'\ell}$$

Furthermore, any function $-1 \leq x \leq 1$ can be expanded in terms of them in a Legendre series representation

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x)$$

$$A_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} f(x) P_\ell(x)\, dx$$

Graphs



$P_\ell(x)$ has ℓ zero's in interval $-1 \leq x \leq 1$.

Other useful recursion formulas

$$\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx} - (2l+1)P_l = 0$$

$$\Rightarrow \quad (l+1)P_{l+1} - (2l+1)xP_l + lP_{l-1} = 0$$

$$\Rightarrow \quad \frac{dP_{l+1}}{dx} - x\frac{dP_l}{dx} - (l+1)P_l = 0$$

$$\Rightarrow \quad (x^2-1)\frac{dP_l}{dx} - lxP_l + lP_{l-1} = 0.$$

Can use such formulas to do computations WLOG $l' \geq l$.

$$I_1 = \int_{-1}^{1} xP_l(x)P_{l'}(x)dx$$

$$= \frac{1}{2l+1}\int_{-1}^{1} P_{l'}(x)\left[(l+1)P_{l+1} + lP_{l-1}(x)\right]dx$$

$$= \begin{cases} \dfrac{2(l+1)}{(2l+1)(2l+3)} & l' = l+1 \\[2ex] \dfrac{2l}{(2l-1)(2l+1)} & l' = l-1 \end{cases}$$

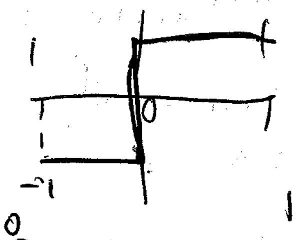
Such formulas are basis of selection rules in Q.M.

$$I_2 = \int_{-1}^{1} x^2 P_l(x)P_{l'}(x)dx = \begin{cases} \dfrac{2(l+1)(l+2)}{(2l+1)(2l+3)(2l+5)} & l' = l+2 \\[2ex] \dfrac{2(2l^2+2l-1)}{(2l-1)(2l+2)(2l+3)} & l' = l. \end{cases}$$

~~More properties~~

Example

Consider $f(x) = \begin{matrix} +1 & \text{for } x>0 \\ -1 & \text{for } x<0 \end{matrix}$



$$A_\ell = \frac{2\ell+1}{2} \int_0^1 P_\ell(x)\,dx - \int_{-1}^{0} P_\ell(x)\,dx$$

Notice $P_\ell(x)$ is odd for ℓ odd and even for even ℓ.
So only odd values contribute

$$A_\ell = 0 \qquad \ell\text{-even}$$
$$A_\ell = (2\ell+1)\int_0^1 P_\ell(x)\,dx \qquad \ell \text{ odd}$$

To evaluate

$$A_\ell = (2\ell+1)\int_0^1 \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

$$\cancel{= \frac{(2\ell+1)}{2^\ell \ell!}\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell \Big|_0^1}$$

$$\cancel{\frac{d^{\ell-2}}{dx^{\ell-2}}\ell(x^2-1)^{\ell-1}2x}$$

~~It's clear all terms w~~

Can do crazy recursion stuff using Rodriguez formula

I (In the modern world use mathematica)

$$A_\ell = \frac{(2\ell+1)}{2^\ell \ell!}\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell \Big|_0^1$$

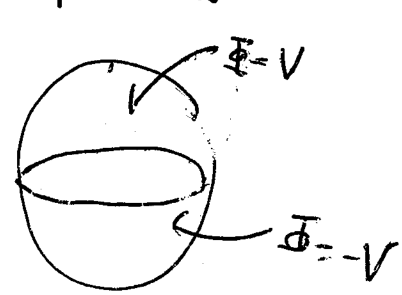
$$= \left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(2\ell+1)(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!}$$

$$f(x) = \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) - \ldots$$

Example

Consider two hemispheres at equal and opposite potentials

$$V(\theta) = \begin{cases} V & 0 \le \theta \le \frac{\pi}{2} \\ -V & \frac{\pi}{2} < \theta \le \pi \end{cases}$$



Find potential inside sphere

From seperation of variables have $k=0$, $m=0$ so general solution is just

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta)$$

Potential finite at $r=0$ so $B_l = 0$.

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

We also have

$$V(\theta) = \sum_{l=0}^{\infty} \underbrace{A_l a^l}\, P_l(\cos\theta)$$

This is Legendre Series so

$$A_l = \frac{2l+1}{2a^l} \int_0^{\pi} V(\theta) P_l(\underbrace{\cos\theta}_{x}) \underbrace{\sin\theta\, d\theta}_{dx}$$

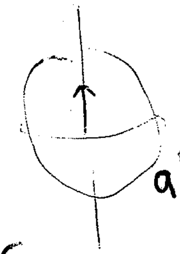
from a^l multiplying

When $V(\theta) = \begin{cases} +V & (0 \le \theta \le \pi/2) \\ -V & (\pi/2 < \theta \le \pi) \end{cases}$ coefficients proportional to example above

so

$$\Phi(r,\theta) = V\left[\frac{3}{2}\left(\frac{r}{a}\right)P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \frac{11}{16}\left(\frac{r}{a}\right)^5 P_5(\cos\theta) + \cdots\right]$$

Potential outside sphere obtained by $\left(\frac{r}{a}\right)^{\ell} \to \left(\frac{a}{r}\right)^{\ell+1}$.



Notice on z-axis $P_\ell(\cos\theta) = (-1)^\ell$ so potential at $\Phi(z=r)$ is related to general potential by multiplying Coefficients A_ℓ, B_ℓ by $P_\ell(\cos\theta)$. Exploit this later ~~to develop Green's funcs~~ to develop Green's Functions

We now consider the more general case of $m \neq 0$

Recall that using seperation of variables, we ~~we~~ we arrived at the associate Legende equation for $x = \cos\theta$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP(x)}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P(x) = 0$$

We saw that for $m=0$ the solutions were ~~Legendre~~ Legendre polynomials. Furthermore requiring $P(x)$ to be finite at $x = \pm 1$ forced ℓ to be a 0 or $\mathbb{Z}^+$

Using same kind of arguments, It can be shown that in order for $P_\ell^m(\cancel{\cos\theta})$ to be finite on $-1 \leq x \leq 1$, $\ell = 0, 1, 2, \ldots$ and $m = -\ell, -(\ell-1), \ldots, \ell-1, \ell$.

In particular, one can show

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x)$$

$$= \frac{(-1)^m}{2^\ell \ell!} (1-x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell$$

Furthermore can show

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$$

These Associated Legendre functions are orthognal.

$$\int_{-1}^{1} P_{\ell'}^m(x) P_\ell^m(x) = \frac{2}{(2\ell+1)} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell\ell'}$$

More generally, we can consider $(\theta \phi)$ solution together to form <u>Spherical Harmonics</u>

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) e^{im\phi}$$

These are the natural basis for spheres and have many nice properties:

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y^*_{lm}(\theta,\phi)$$

$$\int d\phi \int d\theta \sin\theta \; Y^*_{l',m'}(\theta,\phi)\, Y_{lm}(\theta,\phi) = \delta_{ll'}\,\delta_{m'm}$$

One also has a completeness relation and expansion.

Any function

$$g(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta,\phi)$$

$$A_{lm} = \int d(\cos\theta)\, d\phi \;\; Y^*_{lm}(\theta,\phi)\, g(\theta,\phi)$$

Thus, a general solution to Laplace's equation is

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} r^{l} + B_{lm} r^{-(l+1)}\right] Y_{lm}(\theta,\phi)$$

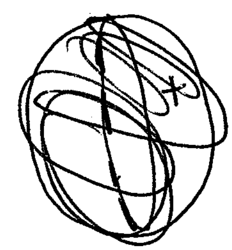
and to Helmhotz equation is

$$\Psi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} j_l(k\rho) + B_{lm} n_l(k\rho)\right] Y_{lm}(\theta,\phi)$$

Let us get some intuition for spherical Harmonics and associated Legendre Polynomials. and Spherical Harmonic

$Y_{\ell m}$

nodal lines → $= \partial$ m great circles through poles and $\ell - m$ circles equal latitude



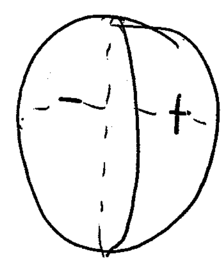
$\ell = 0 \quad Y_{00} = \frac{1}{\sqrt{4\pi}}$

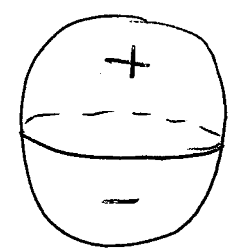
$\ell = 1 \quad Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi}$

$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$

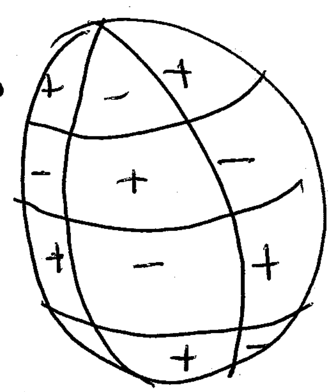
$\ell = 2 \quad \begin{cases} Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta \, e^{i2\phi} \\ Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta \, e^{i\phi} \end{cases}$

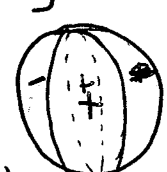
$Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$

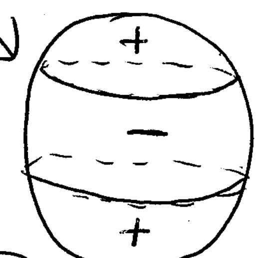


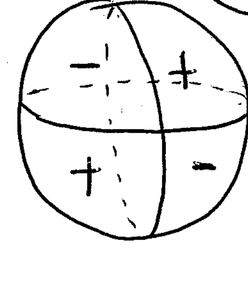


$\ell = 5$
$m = 2 \quad Y_{52}$ →



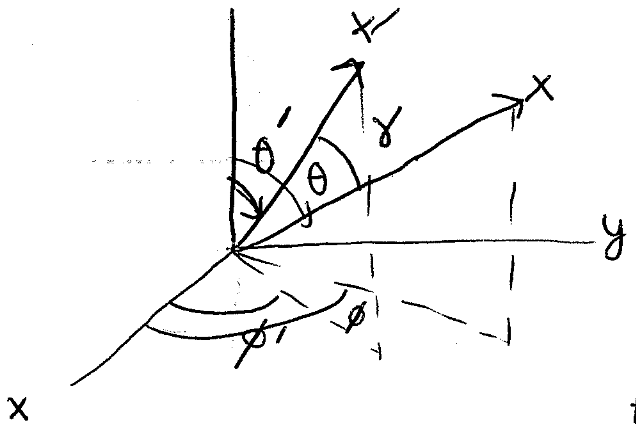






A very important result is Addition Theorem for spherical Harmonics.

Consider



We will use it to find expression for

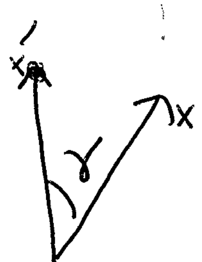
$$\frac{1}{|\vec{X}-\vec{X}'|}$$ in terms of spherical Harmonics

In particular, one can show

Addition Theorem
$$P_l(\cos\gamma) = \frac{4\pi}{(2l+1)} \sum_{m=-l}^{m=l} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

(See Arfken)

Rotate so z-axis is along x' and consider potential due to point charge at North pole



$$\Phi = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-(l+1)}) P_l(\cos\gamma)$$

(Except at $r=0$)

So now we also know from Basic E.M.

$$\Phi = \frac{q}{4\pi\epsilon|X-X'|} = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-(l+1)}) P_l(\underbrace{\cos\gamma}_{1})$$

So now consider when $\gamma = 0$ and we are on z-axis

If $|X| > |X'|$ then we know $\frac{1}{|r-r'|}$ ~ which has series expansion $\frac{1}{r_>} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>}\right)^l$ where $r_<$ ($r_>$) is smaller (larger) of $|X|$ and $|X'|$

Thus, we get coefficients $A_\ell = r'^{\ell+1}$ and $B_\ell = 0$ when $r' > r$ and $A_\ell = 0$ and $B_\ell = r'^\ell$ when $r > r'$.

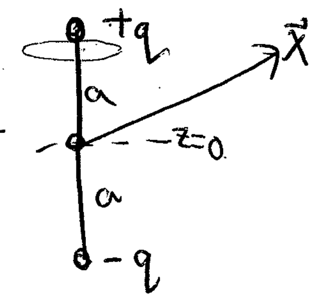
So that
$$\frac{1}{|x-x'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)$$

Plugging in Addition theorem gives
$$\frac{1}{|x-x'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi).$$

So ~~we have sh~~ to get an idea for how all this works – Let us do a detailed example

Example

Consider two point charges q and $-q$ ← We take limit $a \to 0$ with $p/2$ fixed $qa \equiv p/2$ This is dipole moment



Lets find the electrostatic potent in Spherical harmonics

$$\Phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{x}-a|} - \frac{1}{|\vec{x}+a|}\right)$$ ← This problem has azimuthal symmetry so must independent of ϕ

$$= \frac{q}{\epsilon_0} \sum_{\ell m} \frac{1}{(2\ell+1)} \left(\frac{r_<^\ell}{r_>^{\ell+1}}\right) \left[Y^*_{\ell m}(0,\phi') - Y^*_{\ell m}(\pi,\phi') \right] Y_{\ell m}(\theta,\phi)$$

hence only $m=0$ contributes

$$= \frac{q}{\epsilon} \sum_{\ell} \frac{1}{(2\ell+1)} \left(\frac{r_<^\ell}{r_>^{\ell+1}}\right) \left[Y^*_{\ell 0}(0,\phi') - Y^*_{\ell 0}(\pi,\phi') \right] Y_{\ell 0}(\theta,\phi)$$

Since

$$Y_{\ell 0}(0,\phi') = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(1) = \sqrt{\frac{2\ell+1}{4\pi}}$$

$$Y_\ell(\pi,\phi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(-1) = (-1)^\ell \sqrt{\frac{2\ell+1}{4\pi}}$$

So only odd ℓ contribute

$$= \frac{q}{4\pi\epsilon_0} \sum_{\ell \text{ odd}} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell 0}(\theta,\phi) = \frac{q}{2\pi\epsilon_0} \sum_{\ell \text{ odd}} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta)$$

$$= \frac{q}{2\pi\epsilon_0} \sum_{k=0}^{\infty} \frac{r_<^{2k+1}}{r_>^{2k+2}} P_{2k+1}(\cos\theta)$$ exactly what you expect

So now let us take dipole limit. We are interested in limit $a \to 0$ holding $qa = \frac{p}{2}$ so function is $a = r_<$ and $r = r_>$

$$\Phi = \frac{\boxed{qa}}{2\pi\epsilon_0 r^2} \sum_{k=0}^{\infty} \left(\frac{a}{r}\right)^{2k} P_{2k+1}(\cos\theta)$$

See every term except $k=0$ disappears

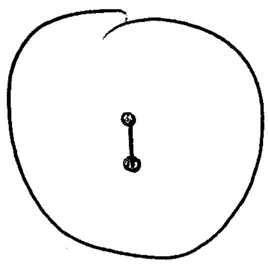
$$\Phi = \frac{p}{4\pi\epsilon_0} \frac{1}{r^2} P_1(\cos\theta) = \frac{p}{4\pi\epsilon_0} \frac{P_1(\cos\theta)}{r^2}$$

This is the dipole moment (potential)

Look at Electric field $\quad \nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}$

$$+\frac{p}{4\pi\epsilon_0}\left[\frac{1}{r^3} P_1(\cos\theta)\hat{r} + \frac{\hat{\theta}}{r^3}\frac{\partial P_1(\cos\theta)}{\partial\theta}\right] =$$

Example Suppose now dipole moment is surrounded by grounded sphere. What is potential inside shell.



Well we can use the superposition principle to solve this problem.

Need solution to add to last solution

$$\nabla^2 \psi = 0 \quad \text{with} \quad \psi = 0 \quad \text{at } r = a$$

General Solution

$$\psi = \sum_{l} A_l r^l P_l(\cos\theta)$$

Now sum of two solutions is

$$\frac{p}{4\pi\epsilon_0} \frac{P_1(\cos\theta)}{r^2} + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi(r=a) = 0.$$

So have

$$\frac{p}{4\pi\epsilon_0} \frac{P_1(\cos\theta)}{a^2} + \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = 0$$

Since orthonormal only $l=1$ has non zero coeffient

$$A_1 = \frac{-p}{4\pi\epsilon_0 a^3} \Rightarrow \Phi = \frac{p}{4\pi\epsilon_0}\left(\frac{1}{r^2} - \frac{r}{a^3}\right)\cos\theta$$

You will do some hard-core B.V. problems for ~~H~~.W.

A concise relationship between Group theory + Spherical harmonics can be found in

This ~~b~~concludes our overview of seperation by variables.

Integral Transforms

For PDE's often useful to use integral transforms:

Fourier transform

Laplace transform $\longrightarrow$ Diffusion

Let $f(t)$ defined for $t \geq 0$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)$$

$$s = \sigma + i\omega$$

Think of Inverse - Laplace transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\, ds$$

Properties:

Linearity

$t f(t) \longrightarrow -F'(s)$

$t^n f(t) \rightarrow (-1)^n F^{(n)}(s)$

$\frac{f(t)}{t} \Rightarrow \int_s^\infty F(\sigma)\, d\sigma$

$f \ast g = \int_0^t g(t) g(t-\tau) \longrightarrow F(s)G(s)$

time shift

$f(t-a)U(t-a) \longrightarrow e^{-as} F(s)$

Basis of Generating Functionals in probability.