Partial Differential Equations

We will spend the remaining 5-6 weeks of the class discussing PDE's – with a special focus on electrodynamics, We start by giving a general intro to PDE's. We will especially be concerned with Second order PDE's. The are unbiqitous in physics.

Examples of PDE's:

1) Laplace's equation $\nabla^2\psi = 0$.

- occurs in Electrostatics and Magnetostatics.

$$\nabla \cdot E = \rho/\varepsilon_0 \quad \text{In absence of charge } 0$$

$$E = -\nabla\Phi$$

$$\Longrightarrow \nabla^2\Phi = 0$$

2) Poisson's Equation (Inhomogenous equation)

$$\nabla^2\psi = -\rho/\epsilon_0$$

3) Wave (Hemholz) equation + time-independent Diffusion equation)

$$\nabla^2\psi \pm k^2\psi = 0$$

- Elastic waves in Solids, membranes, ect.
- E.M. waves

(Do after 4 + 5)

Example

$$\nabla \times \nabla \times E = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

0 in absence of charge

$$\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}[\nabla \times B] = -\frac{\partial}{\partial t}\left[\mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right]$$

$\vec{J} = \sigma \vec{E}$ (Ohm's law)

$$= -\sigma\mu_0 \frac{\partial \vec{E}}{\partial t} - \mu_0\epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \quad \nabla^2 \vec{E} = \sigma\mu \frac{\partial \vec{E}}{\partial t} + \epsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2}$$

If we let $\vec{E} = \vec{E}_0 e^{i\omega t}$ then

$$\frac{\partial \vec{E}}{\partial t} = -i\omega E_0 e^{-i\omega t} = -i\omega \vec{E} \qquad \frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E}$$

$$\nabla^2 \vec{E} = -\omega^2 \epsilon\mu\left(1 + \frac{\sigma i}{\omega\epsilon}\right)\vec{E}$$

$$= -k^2 \vec{E} \qquad k^2 = \omega^2 \mu\epsilon\left(1 + \frac{\sigma i}{\omega\epsilon}\right)$$

Complex wave equation

$$\boxed{\nabla^2 \vec{E} = -k^2 \vec{E}}$$

When $\sigma = 0$. Ohm's Law so ~~no~~ currens

[$k^2 = \omega^2 \mu\epsilon$
this is hemhotz equation]

When $\frac{\sigma}{\epsilon} \to \infty$ this is almost diffusion equation!

④ Time Dependent Diffusion Equation

$$\nabla^2 \psi = \frac{1}{\kappa}\frac{\partial \psi}{\partial t}$$

$$\kappa = \frac{K}{C\rho} = \frac{\text{thermal conductivity}}{(\text{specific heat}) \times \text{density}}$$

⑤ Schrondiger Equaton

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(x)\psi = i\hbar\frac{\partial \psi}{\partial t}.$$

Do example on page 2

Can make some general distinctions.

First homogenous vs non-homogenous equation.

Homogenous equations — ψ is a solution so is any multiple of ψ.

Eg.

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

$$\nabla^2\psi = 0.$$

Also have Inhomogeonous equation in presence of applied "forces" or "sources"

charge density

driving force on string.

$$\nabla^2\Phi = -\rho/\epsilon_0 \qquad \frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -\frac{1}{T}f(x,t)$$

A problem can also become inhomogenous due to boundary conditions so multiples of ψ are not solution.

Example vibrating string which the end $x=0$ prescribed to move in a definite way ; $\psi(0,t) = g(t)$.

General Discussion

We now discuss PDE's in general focusing on what boundary conditions ensure uniqueness of solutions

We consider case where we have two variables.

3 Common types of boundary conditions:

1) Dirchelet conditions: ψ is specified at each point on boundary
2) Neumann Conditions: $(\nabla\psi)_n$ the normal component of the gradient of ψ is specified at each point of the bounday
3) Cauchy Conditions: ψ and $(\nabla\psi)_n$ are specified at each point of the boundary

Things are much more complicated than for ODE's. Certain kinds of PDE's only are well defined for certain kind of B.C.

For an ODE of second order, the specification of ψ and ψ' at boundary x_0 together with DEQ completely determine ψ close to x_0

$$a\psi''(x_0) \pm b\psi'(x_0) + c$$

Take higher derivatives by taking derivatives

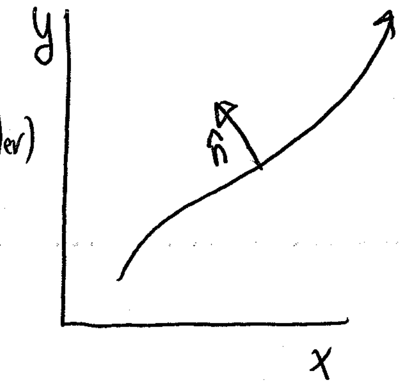
$$\psi^n(x_0)$$

thus solution exists.

We now consider PDEs.

Suppose boundary curve is described parametrically by

$x = x(s)$ and $y = y(s)$.



Assume we know $\psi(s)$ and $N(s)$ (normal der) along boundary

$$\hat{n} = \left(-\frac{dy}{ds}, \frac{dx}{ds}\right)$$

$$N(s) = -\frac{\partial\psi}{\partial x}\frac{dy}{ds} + \frac{\partial\psi}{\partial y}\frac{dx}{ds}$$

$$\frac{d\psi(s)}{ds} = \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial s}$$

$\Rightarrow$ $\frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi}{\partial y}$ can be solved for and these are unique.

Now look at second derivatives

$$\frac{\partial^2\psi}{\partial x^2}, \quad \frac{\partial^2\psi}{\partial x\partial y}, \quad \frac{\partial^2\psi}{\partial y^2}$$

$$\frac{d}{ds}\frac{\partial\psi}{\partial x} = \frac{\partial^2\psi}{\partial x^2}\frac{dx}{ds} + \frac{\partial^2\psi}{\partial x\partial y}\frac{dy}{ds}$$

$$\frac{d}{ds}\frac{\partial\psi}{\partial y} = \frac{\partial^2\psi}{\partial x\partial y}\frac{dx}{ds} + \frac{\partial^2\psi}{\partial y^2}\frac{dy}{ds}$$

also have PDE

$$A\frac{\partial^2\psi}{\partial x^2} + 2B\frac{\partial^2\psi}{\partial x\,\partial y} + C\frac{\partial^2\psi}{\partial y^2} = f\left(x, y, \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right)$$

Can solve for second derivatives unless

$$\begin{vmatrix} \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & \frac{dx}{ds} & \frac{dy}{ds} \\ A & B & C \end{vmatrix} = 0$$

or

or

$$A\left(\frac{dy}{ds}\right)^2 - 2B\left(\frac{dx}{ds}\right)\left(\frac{dy}{ds}\right) + C\left(\frac{dx^2}{ds^2}\right) = 0.$$

The two directions in (x, y) plane determined by this equation are so-called <u>characteristic</u> directions

Curves in the xy-plane whose tangents at each point lie along characteristic directions are called <u>characteristics</u> of PDE.

Along characteristics second derivatives are not specified. Same for higher order derivatives.

It is clear that PDE's will have qualitatively different behaviors depending of characteristic equations.

Switch to coordinate system: (ξ, η) along characteristics. $\rightarrow$ satisfy

~~Classify differential Equations~~

~~$\xi(x,y) = \text{const}$, $\eta(x,y) = \text{const}$~~

$$A\left(\frac{d\xi}{dx}\right)^2 - 2B\left(\frac{d\xi}{dx}\right)\left(\frac{d\xi}{dy}\right) + C\left(\frac{d\xi}{dy}\right)^2 = 0$$

$$A\left(\frac{d\eta}{dx}\right)^2 - 2B\left(\frac{d\eta}{dx}\right)\left(\frac{d\eta}{dy}\right) + C\left(\frac{d\eta}{dy}\right)^2 = 0,$$

Then have three possibilities

$D = ac^2 - b^2$

1) $D < 0$ (Two solutions ξ, η are independent)

<u>These are hyperbolic Diff eq</u>

<u>Example</u> Wave equation

2) $D = 0$ (only one

<u>These are parabolic Diff eq</u>

<u>Example</u> Diffussion Equation

3) If $D > 0$ Elliptic equations

Example: Poisson Equation

$$c^{-2}\frac{\partial}{\partial t^2} + \nabla^2$$

Example 1 Consider wave equation

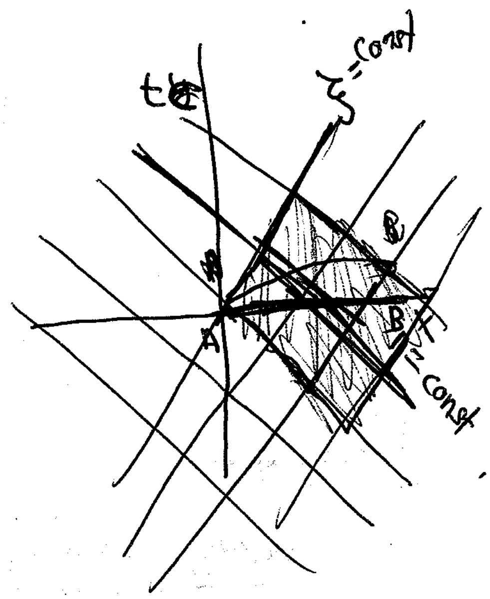
$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} = 0 \qquad D < 0 \quad \text{hyperbolic}$$

$\Rightarrow$ Characteristic equation

$$\left(\frac{dt}{ds}\right)^2 - \frac{1}{c^2}\left(\frac{dx}{ds}\right)^2 = 0$$

or

$$\left(\frac{dx}{dt}\right)^2 = c^2$$



$$\Rightarrow \quad (x - ct) = \xi = \text{Constant}$$
$$(y - ct) = \eta = \text{constant}$$

In this natural coordinates equation becomes

$$\frac{\partial^2\psi}{\partial\xi\,\partial\eta} = 0. \longleftarrow \text{normal form}$$

$$\left(\frac{d}{dx} - \frac{1}{c}\frac{d}{dt}\right)\left(\frac{d}{dx} + \frac{1}{c}\frac{d}{dt}\right)\psi = 0$$

General Solution $\psi = F(x+ct) + G(x-ct)$

Say we know ψ and $N(x) = c^{-1}\partial\psi/\partial t$ along segment AB where they have values ~~F(x) and~~ ~~F(x) and g(x)~~. $\psi = F(x) + G(x)$ $\quad N(x) = F'(x) - g'(x) = \frac{1}{c}\frac{\partial\psi}{\partial t}$

Solve everywhere

$$F(x) = \frac{1}{2}\psi(x) - \frac{1}{2}\int\frac{\partial\psi}{\partial t}\,dx \qquad g(x) = \frac{1}{2}\psi(x) + \frac{1}{2}\int\frac{\partial\psi}{\partial t}$$

Example 2 Consider Laplaces equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

Charcateristic equation

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 0$$

$$\Rightarrow \left(\frac{dy}{dx}\right) = \pm i$$

$$\Rightarrow \left.\begin{array}{l} y + ix = \xi \\ y - ix = \eta \end{array}\right]$$

These are characteristics

General Solution $\psi = F(x+iy) + G(x-iy)$

$$\mathcal{L} = \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} = 0.$$

This is normal form

~~Example 3~~

Boundary Conditions

These different kind of differential equations are well defined for different kinds of boundary conditions.

~~For example, consider~~

This can be summarized in a chart

B.C	Elliptic	Hyperbolic	Parabolic
	Poisson	Wave	Diffusion equation
Cauchy			
Open surface	Unphysical results (instability)	Unique, stable solution	too restrictive
closed surface	Too restrictive	Too restrictive	too restrictive
Dirchlet			
Open Surface	Insufficient	Insufficient	Unique stable solution in one direction
closed surface	Unique, stable Solution	Solution not unique	Too restrictive
Neumann			
Open Surface	Insufficient	Insufficient	Unique Stable solution in one direction
Closed surface	Unique, stable solution	Solution not unique	Too restrictive

Seperation of Variables → Try to turn PDE into ODE's

Example

Consider wave equation

$$\nabla^2 \psi - \frac{1}{c^2}\frac{d\psi}{dt^2} = 0$$

Look for solution of form $\psi(x,t) = X(x)T(t)$

Substituting this into (8-21)

$$\nabla^2 X(x)T(t) - \frac{1}{c^2}\frac{\partial^2 T(t)}{\partial t^2} X(x) = 0$$

$$\Rightarrow \frac{\nabla^2 X(\vec{x})}{X} = \frac{1}{c^2}\frac{1}{T}\frac{d^2 T}{dt^2}$$

This is function only of X

This is function only of T

So they must equal constant $= -k^2$

$$\nabla^2 X = -k^2 X$$] Hemholtz equation

$$\frac{d^2 T}{dt^2} = -\omega^2 T \qquad \omega = kc$$

$$\Rightarrow T = \left\{ \begin{matrix} \sin \omega t \\ \cos \omega t \end{matrix} \right\}$$ ← Any linear combination of these

Look at Hemholtz equation

$$\nabla^2 X = -kX^2$$

Then try $X = X(x)Y(y)Z(z)$.

Equation becomes

$$\underbrace{\frac{1}{X(x)}\frac{d^2X}{dx^2}} = -k^2 - \frac{1}{Y}\frac{d^2Y}{dY} - \frac{1}{Z}\frac{dZ}{dZ}$$

again must equal constant $\equiv -\ell^2$

$$\frac{1}{X}\frac{dX}{dx^2} = -\ell^2$$

Continue

$$-k^2 + \ell^2 - \frac{1}{Z}\frac{d^2Z}{dZ^2} = \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-m^2}$$

$$\frac{1}{Z}\frac{dZ^2}{dZ^2} = -k^2 + l^2 + m^2 - n^2$$

Consider

So $\quad k^2 = l^2 + m^2 + n^2$

$$X(\vec{x}) = X_\ell(x)\, Y_m(y)\, Z_n(z)$$

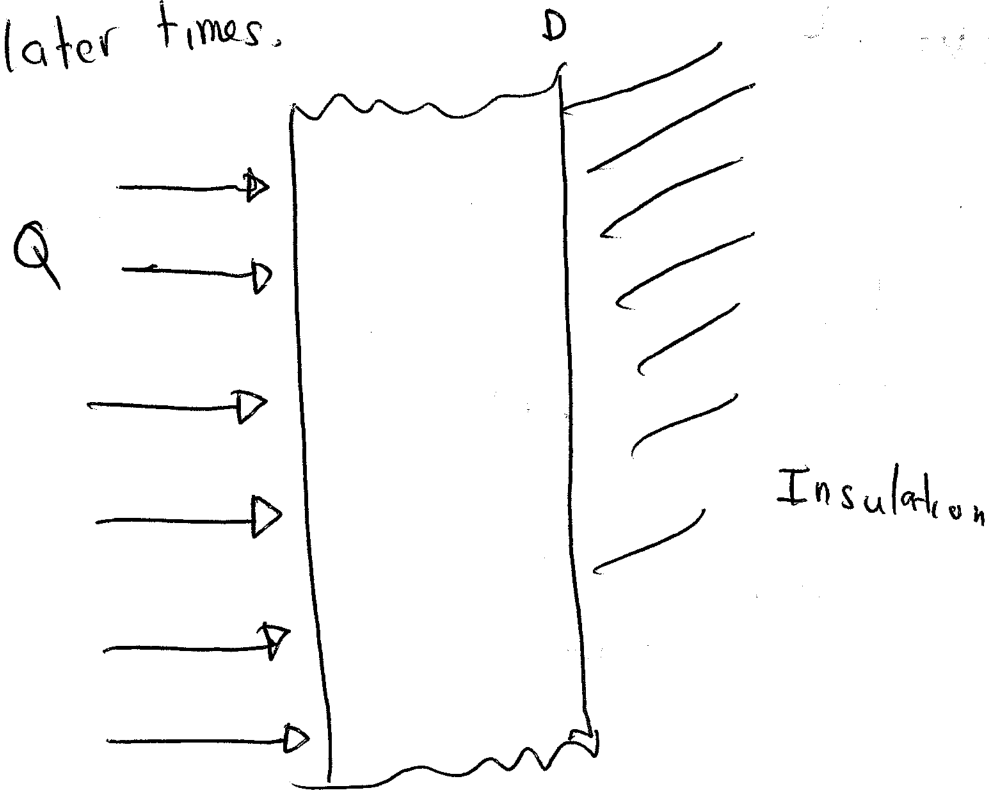
So general solution

$$\psi = \sum_{\ell, m, n} X_\ell(x) Y_m(Y) Z_n(z) \left[a_{\ell mn} \sin \omega_{\ell mn} t + b_{\ell mn} \cos \omega_{\ell mn} t\right]$$

So now need to specify boundary conditions by choosing $a_{\ell mn}$ and $b_{\ell mn}$ appropriately.

Example ~~Triangular Drum~~

Consider an infinite heat conducting slab of thickness D with one surface $(X=D)$ insulated. Initially, $t=0$, the $T=0$ and heat is supplied at a constant rate Q calories / sec cm^2 at $x=0$. Find $T(x,t)$ at later times.



Solve diffusion equation

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{\kappa}\frac{\partial T}{\partial t} = 0 \qquad \kappa = \frac{K}{C\rho}$$

thermal Conductivity

After long time we expect temperature to rise linearly with time.

Try $\quad T_p = U(x) + \alpha t,$] (Separation of variables)

Thus, plugging equation

$$\frac{\partial^2 U}{\partial x^2} - \frac{\alpha}{K} = 0$$

$$\Rightarrow \quad U(x) = \frac{1}{2}\frac{\alpha}{K}x^2 + \alpha x + b$$

B.C.

$$-\underbrace{KU'(0)}_{\text{(Ohms Law)}} = Q \qquad a = -\frac{Q}{K}$$

$$U'(D) = 0$$

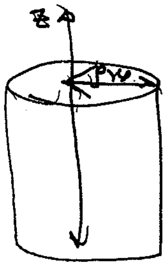
$$U(x) = \frac{1}{2}\frac{Q}{DK}(x-D)^2 \qquad \alpha = \frac{Q\cancel{K}}{KD} = \frac{Q}{C\rho D}$$

$$\boxed{T_p = \frac{1}{2}\frac{Q}{KD}(x-D)^2 + \frac{Q}{C\rho D}t}$$

These kind of solution that grow linearly with time are often used in Hamilton-Jacobi' equations.

For the remainder of the class we will talk about how to use seperation of variables to solve Hemholtz equation in spherical and cylindrical coordinates.

Let us now consider Helmholtz equation is cylinderical coordinates



$$\nabla^2\psi(\rho,\varphi,z) + k^2\psi(\rho,\varphi,z) = 0$$

The Laplacian in cylindrical coordinates becomes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0$$

We can again use seperation of variables

$$\psi(\rho,\varphi,z) = R(\rho)\,\Phi(\varphi)\,Z(z)$$ (since orthonormal system)

Substituting

$$\frac{\Phi Z}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{RZ}{\rho^2}\frac{d^2\Phi}{d\varphi^2} + R\Phi\frac{d^2Z}{dz} + k^2R\Phi Z = 0$$

Divide by $P\Phi Z$

$$\underbrace{\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\varphi^2}}_{\text{independent of } z} + k^2 = \underbrace{-\frac{1}{Z}\frac{d^2Z}{dz^2}}_{\text{depends only on } z}$$

So equals a constant $= -\ell^2$

So the Z equations becomes

$$\boxed{\frac{d^2 Z}{dz^2} = \ell^2 Z} \rightsquigarrow \text{Solution} \begin{cases} \ell = 0 \\ \quad Z = A + Bz \\ \ell \neq 0 \\ \quad Z = Ae^{\ell z} + Be^{-\ell z} \end{cases}$$

and

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{\rho^2 \Phi}\frac{d^2\Phi}{d\varphi^2} + k^2 = -\ell^2$$

Now muliply by ρ^2.

$(k^2+\ell^2)$

$$\underbrace{\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + n^2\rho^2}_{\text{Depends on } \rho} = \underbrace{-\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}}_{\text{Depends on } \varphi \,\rightsquigarrow\, m^2}$$

$$\boxed{\frac{d^2\Phi}{d\varphi^2} = -m^2\Phi} \rightsquigarrow \text{Solution} \begin{cases} m = 0 \\ \Phi = C + D\varphi \\ m \neq 0 \\ \Phi = C\cos m\varphi + D\sin m\varphi \end{cases}$$

The equation for P then becomes

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + (n^2\rho^2 - m^2)P = 0$$

Two cases $\begin{cases} n = 0 & \text{This is a straigh forward equation} \\ n \neq 0 & \text{This is "Bessel's Differential Equation"} \end{cases}$

Consider first

Laplace's equation with ~~no~~ z dependence linear

$n=0$ $(k=0,\ \ell=0)$ Then

$$\rho \frac{d}{d\rho}\left(\rho \frac{dP}{d\rho}\right) = m^2 P$$

$$\begin{cases} \nu=0 & R(\rho) = E + F\ln\rho \\ \nu\neq 0 & R(\rho) = E\rho^{\nu} + F\rho^{-\nu} \end{cases}$$

Laplace's equation In ~~no~~ dependence on z and φ linear

Laplace's equation with z dependence linear.

Now consider general case:

$$\rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + (n^2\rho^2 - \nu^2) P = 0.$$

Useful to change parameters to $x = n\rho$

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0$$

Solutions to this are Bessel functions and we review some general properties

Assume power series solutions.

$$R(x) = x^{\alpha}\sum_{j=0}^{\infty} a_j x^j$$

So now plug this into the equation above

$$\sum_{j=0}^{\infty} (j+\alpha)(j+\alpha-1)a_j x^{j+\alpha-2} + (j+\alpha)a_j x^{j+\alpha-2} + a_j x^{j+\alpha} - \nu^2 a_j x^{j+\alpha-2} = 0.$$

Look at term proportional to $x^{\alpha-2}$

$$\alpha(\alpha-1)a_j + \alpha a_j - \nu^2 a_j = 0$$

$$\Rightarrow \alpha^2 = \nu^2$$

$$\Rightarrow \boxed{\alpha = \pm\nu}$$

Now look at terms proportional to $x^{\alpha-1}$

$$(\alpha+1)\alpha a_1 + \alpha a_1 - \nu^2 a_1 = 0$$

Since $\alpha = \pm\nu$

$$(\pm\nu+1)(\pm\nu) \pm \nu a_1 - \nu^2 a_1 = 0$$

$$[\nu^2 \pm \nu - \nu^2] a_1 = 0$$

$$\Longrightarrow \pm\nu a_1 = 0$$

$$\Longrightarrow \boxed{a_1 = 0}$$

For $x^{j+\alpha-2}$ term for $j \geq 2$. Notice relate

$$(j+\alpha)(j+\alpha-1)a_j + (j+\alpha)a_j + a_{j-2} - \nu^2 a_j = 0$$

$$(j+\alpha)^2 a_j - \nu^2 a_j = -a_{j-2}$$

$$(j^2 \pm 2\underbrace{\nu}_{\alpha} j + \cancel{\nu^2} - \cancel{\nu^2}) a_j = -a_{j-2}$$

$$a_j = \frac{-a_{j-2}}{j(j+2\alpha)}$$

So we have recursion for a_j in terms of a_{j-2}. Since $a_1 = 0$ only odd terms are kept so conventional to rewrite

$$a_{2j} = \frac{-1}{4j(j+\alpha)} a_{2j-2}$$

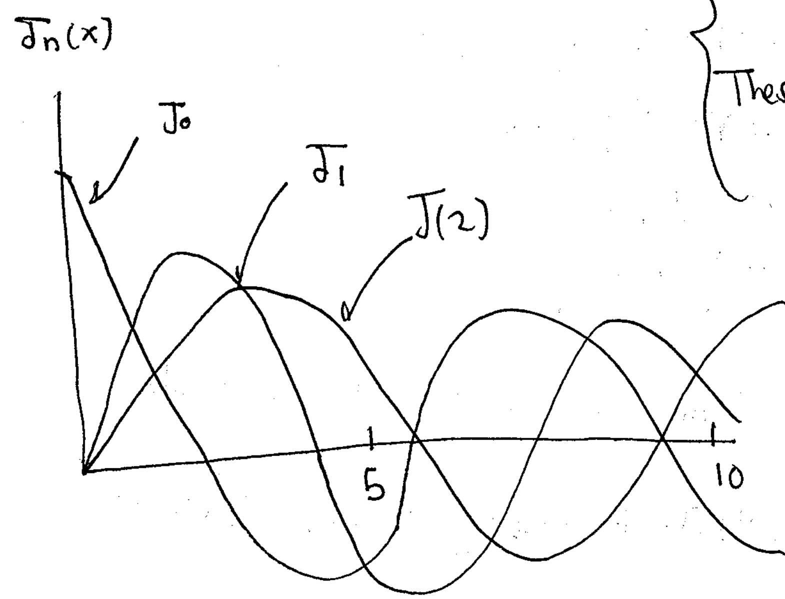
$$a_{2j} = \frac{(-1)^j \Gamma(\alpha+1)}{2^{2j} j! \, \Gamma(j+\alpha+1)} a_0$$

$$\Longrightarrow J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \, \Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j}$$

$$J_{-\nu} = \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \, \Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j}$$

These solutions are called Bessel functions of the first kind.

If ν is <u>not an integer</u>, these are independent.



polynomial functions for small x

These are basically oscillatory functions for large x

Crossover around $x \sim \nu$

Thus have infinite number of roots

For ν integer, see $J_{-m}(x) = (-1)^m J_m(x)$ so not independent!

For integers, replace $J_{\pm\nu}(x)$ by $J_\nu(x)$ and $N_\nu(x)$ [Neumann function or Bessel function of second kind]

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$

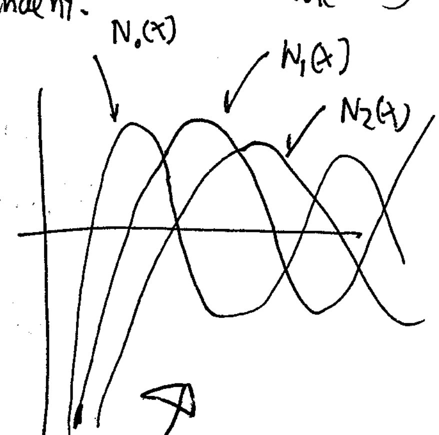
For $\nu \notin \mathbb{Z}$ clearly $J_\nu(x)$ and $N_\nu(x)$ are independent.

However, still independent ($\lim_{\nu\to\mathbb{Z}}$ exist by L'Hopital rule)

Also form Hankel Functions

$$H_\nu^{(1)}(x) = J_\nu(x) + i N_\nu(x)$$

$$H_\nu^{(2)}(x) = J_\nu(x) - i N_\nu(x)$$



[Important Blow up at zero !]

~~Using this equation we can solve Laplace's equation.~~

Laplaces equation

Consider case If we had taken the case $k^2 = 0$ and

$$\frac{d^2Z}{dz^2} = -\ell^2 Z = -n^2 Z \quad (n^2 = \cancel{k^2} + \ell^2 = \ell^2)$$

Then corresponding Bessel equation after transforming to

$x = n\rho$ is

sign changes.

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)R = 0$$

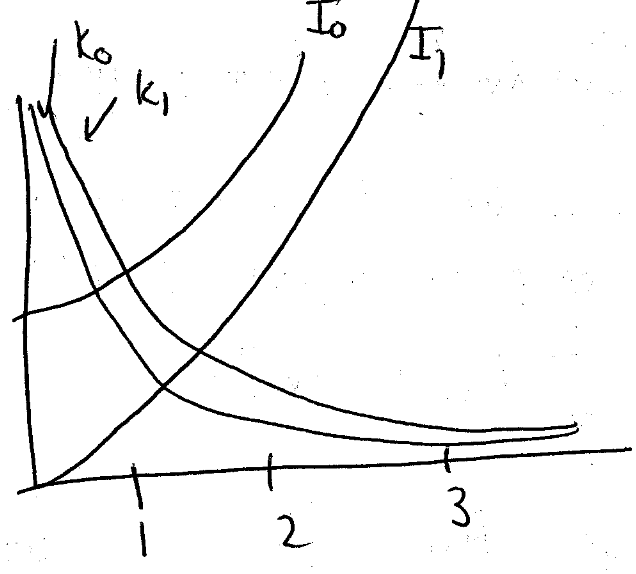
The solutions to these are the modified Bessel functions

$$I_\nu(x) = -i^\nu J_\nu(ix)$$

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(x)$$

I_ν blow up at large x

~~K_ν~~ K_ν blow up at small x.



Bessel Functions all satisfy

$$\Omega_{\nu-1}(x) + \Omega_{\nu+1}(x) = \frac{2\nu}{x}\Omega_\nu(x)$$

$$\Omega_{\nu-1}(x) - \Omega_{\nu+1}(x) = 2\frac{d\Omega_\nu(x)}{dx}$$

An important property that ~~is importa~~ Bessel Functions Satisfy is that they are "orthognal" in some sense (This is the basis of Strum-Louiville theory) which we will re-visit in a few weeks.

Since Bessel functions eventually oscillate, they have infinite number of roots

$$J_\nu(x_{\nu n}) = 0 \qquad n = 1, 2, 3, \cdots$$

So now we show $\sqrt{\rho}\, J_\nu(x_{\nu n}\rho/a)$ for fixed $\nu \geq 0$, $n = 1, 2, \cdots$ for orthognal set on $0 \leq \rho \leq a$.

Namely, they satisfy

$$\int_0^a \rho\, J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right) J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) d\rho = \frac{a^2}{2}\left[J_{\nu+1}(x_{\nu n})\right]^2 \delta_{n'n}$$

(Proof see Jackson Section 3.7 or Arfken)

So any function of ρ or interval $0 \leq \rho \leq a$ can be expanded in

Fourier-Bessel series

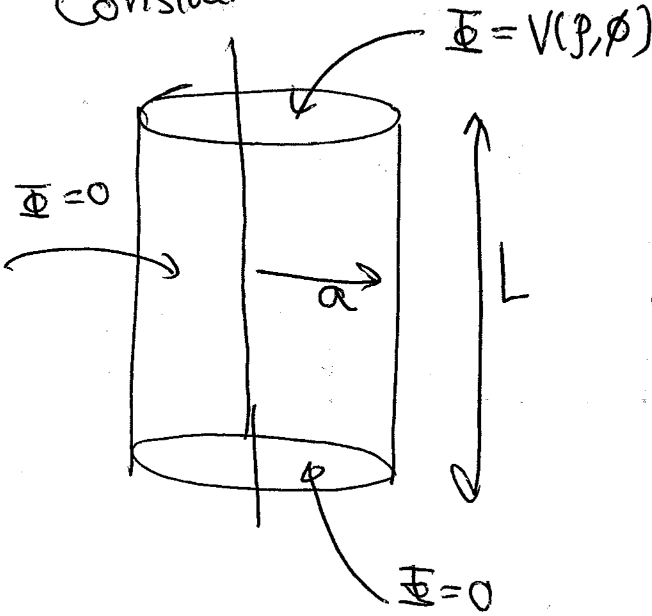
$$f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)$$

$$A_{\nu n} = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})} \int_0^a \rho\, f(\rho)\, J_\nu\left(\frac{x_{\nu n}\rho}{a}\right) d\rho$$

This kind of series is convenient for function that vanish at $\rho = a$ (Homogenous Dirchelet B.C. on a cylinder)

So with all that done lets solve a boundary Value Problem for Electrostatic potential

Consider



$$\nabla^2 \Phi = 0,$$

Since we have symmtery in ϕ we use a seperation of variables as before

$$Z(z) = Ae^{lz} + Be^{lz}$$

$$\Phi(\varphi) = C\cos m\phi + D\sin m\phi$$

($\nu = m$ integer since must be periodic)

$$R(\rho) = E\, J_m(k\rho) + F\, N_m(k\rho)$$

Now look at B.C. We know $\Phi = 0$ at $z=0$.

This implies $A = -B$ so

$$Z(z) = \sinh lz$$

Furthermore, we know that $N_m(k\rho)$ blow up at $\rho = 0$ where potential is finite so $F=0$.

$$R(\rho) = E\, J_m(k\rho),$$ Furthermore, $\Phi(\rho = a) = 0$ so $k = \frac{X_{mn}}{a} = K_{mn}$

So we know that

$$\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(K_{mn}\rho)\sinh(K_{mn}z)\left(A_{mn}\sin m\phi + B_{mn}\cos m\phi\right)$$

At $z=L$ we have

$$V(\rho,\phi) = \sum_{m,n} \sinh(k_{mn}L) J(k_{mn}\rho)\left(A_{mn}\sin m\phi + B_{mn}\cos m\phi\right)$$

This Fourier series in ϕ and Fourier-Bessel in ρ.

So

$$A_{mn} = \frac{2\,\mathrm{cosech}(k_{mn}L)}{\pi a^2 J_{m+1}^2(k_{mn}a)} \int_0^{2\pi} d\phi \int d\rho\, V(\rho,\phi)\rho J_{mn}(\rho)\sin m\phi$$

$$B_{mn} = \quad '' \quad \int_0^{2\pi} d\phi \int d\rho\, V(\rho,\phi)\rho J_m(k_{mn}\rho)\cos m\phi$$

where we have used Fourier-Bessel expansion formula

$$V(\rho,\phi) = \sum_{m,n} \int d\phi \left[\frac{\int d\rho\, \rho\, V(\rho,\phi)}{a^2 J_{m+1}^2(k_{mn}\rho)}\right]^2 J_m(k_{mn}\rho)\sin m\phi\left(\frac{1}{\pi}\right)$$

and Fourier serise

$$f(\phi) = \frac{1}{2}A_0 + \sum_{m=1}^{\infty} A_m \cos(m\phi) + B_m \sin m\phi$$

$$A_m = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos m\phi$$

$$B_m = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\sin m\phi$$