Partial Differential Equations

We will spend the remaining 5-6 weeks of the class discussing PDE's — with a special focus on electrodynamics.

We start by giving a general intro to PDE's. We will especially be concerned with second order PDE's. They are ubiquitous in physics.

Examples of PDE's:

1) Laplace's equation $\nabla^2 \psi = 0$.
   - occurs in Electrostatics and Magnetostatics.

   $\nabla \cdot E = \rho/\varepsilon_0$ \hspace{1cm} \text{In absence of charge} \hspace{.5cm} 0

   $E = -\nabla \psi$

   $\Rightarrow \nabla^2 \psi = 0$

2) Poisson's Equation (Inhomogenous equation)

   $\nabla^2 \psi = -\rho/\varepsilon_0$

3) Wave (Helmholtz) equation + time-independent Diffusion equation

   $\nabla^2 \psi \pm k^2 \psi = 0$

   - Elastic waves in solids, membranes, etc.
   - E.M. waves
Example

$$\nabla \times \nabla \times \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

In absence of charge

$$\nabla \times (-\frac{\partial \mathbf{B}}{\partial t}) = -\frac{\partial}{\partial t} \left[ \nabla \times \mathbf{B} \right] = -\frac{\partial}{\partial t} \left[ \frac{1}{2} \mathbf{J} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\varepsilon_0}{\sigma} \frac{\partial \mathbf{E}}{\partial t} \right]$$

$$= -\sigma \mu \frac{\partial \mathbf{E}}{\partial t} - \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$= 0$$

$$\nabla^2 \mathbf{E} = \sigma \mu \frac{\partial \mathbf{E}}{\partial t} + \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

If we let $$\mathbf{E} = \mathbf{E}_0 \exp(\omega t)$$ then

$$\frac{\partial \mathbf{E}}{\partial t} = -i \omega \mathbf{E}_0 \exp(-\omega t) = -\omega \mathbf{E} \quad \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\omega^2 \mathbf{E}$$

$$\nabla^2 \mathbf{E} = \omega^2 - \omega^2 \varepsilon_0 \mu \frac{(1+\sigma \varepsilon_0 \omega) \mathbf{E}}{\omega}$$

$$= -K^2 \mathbf{E}$$

$$\nabla^2 \mathbf{E} = -K^2 \mathbf{E}$$

$$K^2 = \omega^2 \varepsilon_0 \mu (1+\sigma \varepsilon_0 \omega)$$

When $$\sigma = 0$$, Ohm's Law

$$K^2 = \omega^2 \varepsilon_0 \mu$$

this is Helmholtz equation

When $$\frac{\sigma}{\varepsilon_0} \to \infty$$ this is almost diffusion equation!
(4) **Time Dependent Diffusion Equation**

\[ \nabla^2 \psi = \frac{1}{\kappa} \frac{\partial \psi}{\partial t} \]

\[ \kappa = \frac{K}{C_P \rho} = \frac{\text{thermal conductivity}}{\text{(specific heat)} \times \text{density}} \]

(5) **Schrödinger Equation**

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi + V(x)\psi = i\hbar \frac{\partial \psi}{\partial t} \]

Do example on page 2

Can make some general distinctions.

First **homogenous vs non-homogenous equation**.

Homogenous equations - \( \psi \) is a solution so is any multiple of \( \psi \).

\[ \nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \]

\[ \nabla^2 \psi = 0. \]

Also have **inhomogenous equation** in presence of applied forces or sources.

\[ \nabla^2 \phi = -\rho e_0 \]

\[ \frac{\partial \psi}{\partial x} - \frac{i}{c} \frac{\partial^2 \psi}{\partial t^2} = -\frac{1}{\kappa} f(x,t) \]

A problem can also become inhomogenous due to boundary conditions so multiples of \( \psi \) are not solution.

Example vibrating string which the end \( x=0 \) prescribed to move in a definite way.

\( \psi(0,t) = g(t) \).
General Discussion

We now discuss PDE's in general focusing on what boundary conditions ensure uniqueness of solutions.

We consider the case where we have two variables.

3 Common types of boundary conditions:
1) **Dirichlet conditions**: \( \psi \) is specified at each point on boundary.
2) **Neumann Conditions**: \( (\nabla \psi)_n \), the normal component of the gradient of \( \psi \) is specified at each point of the boundary.
3) **Cauchy Conditions**: \( \psi \) and \( (\nabla \psi)_n \) are specified at each point of the boundary.

Things are much more complicated than for ODE's. Certain kinds of PDE's only are well defined for certain kinds of B.C.

For an ODE of second order, the specification of \( \psi \) and \( \psi' \) at boundary \( x_0 \) together with DEQ completely determine \( \psi \) close to \( x_0 \):

\[
\alpha \psi''(x_0) + \beta \psi'(x_0) + c
\]

Take higher derivatives by taking derivatives:

\( \psi''(x_0) \)

Thus solution exists.
We now consider PDEs.

Suppose boundary curve is described parametrically by
\[ x = x(s) \quad \text{and} \quad y = y(s). \]

Assume we know \( \psi(s) \) and \( N(s) \) (normal derivative) along boundary
\[ \hat{n} = \left( -\frac{dy}{ds}, \frac{dx}{ds} \right) \]
\[ N(s) = -\frac{d\psi}{dx} \frac{dx}{ds} + \frac{d\psi}{dy} \frac{dy}{ds} \]
\[ \frac{d\psi(s)}{ds} = \frac{d^2\psi}{dx^2} \frac{dx}{ds} + \frac{d^2\psi}{dy^2} \frac{dy}{ds} \]
\[ = 0 \quad \frac{d\psi}{dx} \quad \text{and} \quad \frac{d\psi}{dy} \]
\[ \text{can be solved for and these are unique.} \]

Now look at second derivatives
\[ \frac{\partial^2\psi}{\partial x^2}, \quad \frac{\partial^2\psi}{\partial x \partial y}, \quad \frac{\partial^2\psi}{\partial y^2} \]
\[ \text{Can solve for second derivatives unless} \]
\[ \left| \begin{array}{ccc}
\frac{dx}{ds} & \frac{dy}{ds} & 0 \\
0 & \frac{dx}{ds} & \frac{dy}{ds} \\
A & B & C
\end{array} \right| = 0 \]
\[ A \frac{\partial^2\psi}{\partial x^2} + B \frac{\partial^2\psi}{\partial x \partial y} + C \frac{\partial^2\psi}{\partial y^2} = f(x, y, \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}) \]
\[ \text{or} \]
or

\[ A \left( \frac{dy}{ds} \right)^2 - 2B \left( \frac{dx}{ds} \right) \left( \frac{dy}{ds} \right) + C \left( \frac{dx}{ds} \right)^2 = 0. \]

The two directions in \((x, y)\) plane determined by this equation are so-called characteristic directions. Curves in the \(xy\)-plane whose tangents at each point lie along characteristic directions are called characteristics of PDE.

Along characteristics, second derivatives are not specified. Same for higher order derivatives.

It is clear that PDE's will have qualitatively different behaviors depending on characteristic equations. Switch to coordinate system \((\xi, \eta)\) along characteristics.

Classify Differential Equations

Then have three possibilities

\[ D = ac^2 - b^2 \]

1) \(D < 0\) (Two solutions \(\xi, \eta\) are independent)

These are hyperbolic Diff eq.

2) \(D = 0\) (only one)

These are parabolic Diff eq.

Example Diffusion Equation
3) If \( D > 0 \) Elliptic equations

Example 1 Poisson Equation
\[ \nabla^2 \phi = \frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \]

Example 1 Consider Wave equation
\[ \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} = 0 \quad D < 0 \quad \text{hyperbolic} \]

\[ \Rightarrow \quad \text{Characteristic equation} \]
\[ \left( \frac{dt}{ds} \right)^2 - \frac{1}{c^2} \left( \frac{dx}{ds} \right)^2 = 0 \]

or
\[ \left( \frac{dx}{dt} \right)^2 = c^2 \]

\[ \Rightarrow \]
\[ (x - ct) = \xi \quad \text{Constant} \]
\[ (y - ct) = \eta \quad \text{Constant} \]

In this natural coordinates equation becomes normal form
\[ \frac{\partial^2 \psi}{\partial \xi \partial \eta} = 0 \]

\[ \left( \frac{d}{dx} - \frac{i}{c} \frac{d}{dt} \right) \left( \frac{d}{dx} + \frac{i}{c} \frac{d}{dt} \right) \psi = 0 \]

General Solution \( \psi = F(x+ct) + G(x-ct) \)

Say we know \( \psi \) and \( N(x) = c^2 \frac{d^2 \psi}{dt^2} \) along segment AB where they have values everywhere

\[ F(x) = \frac{1}{2} \psi(x) - \frac{1}{2} \int \frac{d\psi}{dx} \, dx \quad g(x) = \frac{1}{2} \psi(x) + \frac{1}{2} \int \frac{d\psi}{dx} \, dx \]
Example 2

Consider Laplace's equation

\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \]

Characteristic equation

\[ \left( \frac{d\lambda}{ds} \right)^2 + \left( \frac{d\eta}{ds} \right)^2 = 0. \]

\[ \Rightarrow \quad \left( \frac{d\eta}{dx} \right)^2 = \pm i \]

\[ \Rightarrow \quad \eta \pm ix = \xi \]

\[ \eta - ix = \eta \]

These are characteristics.

General solution \( \psi = F(x+\xi y) + G(x-\xi y) \)

\[ L = \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} = 0. \] This is normal form.

Example 3

Boundary Conditions

These different kinds of differential equations are well defined for different kinds of boundary conditions.

For example, consider
This can be summarized in a chart:

<table>
<thead>
<tr>
<th>B.C.</th>
<th>Elliptic</th>
<th>Hyperbolic</th>
<th>Parabolic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy open surface</td>
<td>Poisson</td>
<td>Wave</td>
<td>Diffusion equation</td>
</tr>
<tr>
<td>Cloud surface</td>
<td>Unphysical results (Instability)</td>
<td>Unique, stable solution</td>
<td>too restrictive</td>
</tr>
<tr>
<td>Pithchlet, open surface</td>
<td>Too restrictive</td>
<td>Too restrictive</td>
<td>too restrictive</td>
</tr>
<tr>
<td>Neumann open surface</td>
<td>Insufficient</td>
<td>Insufficient</td>
<td>Unique, stable solution in one direction</td>
</tr>
<tr>
<td>Closed surface</td>
<td>Unique, stable solution</td>
<td>Insufficient</td>
<td>Too restrictive</td>
</tr>
<tr>
<td></td>
<td>Solution not unique</td>
<td>Insufficient</td>
<td>Unique, stable solution in one direction</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Too restrictive</td>
</tr>
</tbody>
</table>

Separation of Variables → Try to turn PDE into ODE's

Example

Consider wave equation

\[ ∇^2 \psi = \frac{1}{c^2} \frac{d\psi}{dt^2} = 0 \]

Look for solution of form

\[ \psi(x, t) = X(x) T(t) \]
Substituting this into \((8-21)\)

\[
\nabla^2 X(x) T(t) - \frac{1}{c^2} \frac{\partial^2 T(t)}{\partial t^2} X(x) = 0
\]

\[
\Rightarrow \quad \frac{\nabla^2 X(x)}{X} = \frac{1}{c^2} \frac{1}{T} \frac{\partial^2 T(t)}{\partial t^2} 
\]

This is function only of \(T\)

This is function only of \(X\)

So they must equal constant \(= -k^2\)

\[
\nabla^2 X = -k^2 X \quad \text{Helmholtz equation}
\]

\[
\Rightarrow \quad \frac{\partial^2 T}{\partial t^2} = -\omega^2 T \\
\omega = k c
\]

\[
\Rightarrow \quad T = \left\{ \sin \omega t, \cos \omega t \right\} \\
\text{Any linear combination of these}
\]

Look at Helmholtz equation

\[
\nabla^2 X = -k X^2
\]

Then try \(X = X(x) Y(y) Z(z)\).
Equation becomes
\[ \frac{1}{X(x)} \frac{d^2 X}{dx^2} = -k^2 - \frac{1}{Y} \frac{d^2 Y}{dy^2} - \frac{1}{Z} \frac{d^2 Z}{dz^2} \]

\[ \text{again equal constant } \equiv -l^2 \]
\[ \frac{1}{x} \frac{dX}{dx} = -l^2 \]

Continue
\[ -k^2 + l^2 - \frac{1}{Z} \frac{d^2 Z}{dz^2} = \frac{1}{Y} \frac{d^2 Y}{dy^2} \]
\[ -m^2 \]
\[ \frac{1}{Z} \frac{d^2 Z}{dz^2} = -k^2 + l^2 + m^2 - \eta^2 \]

So
\[ k^2 = l^2 + m^2 + \eta^2 \]
\[ X(x) = X_e(x) Y_m(y) Z_n(z) \]

So general solution
\[ \psi = \sum_{e, m, n} X_e(x) Y_m(y) Z_n(z) [a_{e mn} \sin \omega_{e mn} t + b_{e mn} \cos \omega_{e mn} t] \]

So now need to specify boundary conditions, by choosing \( a_{e mn} \) and \( b_{e mn} \) appropriately.
Example

Consider an infinite heat conducting slab of thickness D with one surface (x=0) insulated. Initially, t=0, the T=0 and heat is supplied at a constant rate $Q$ calories/sec cm² at x=0. Find $T(x,t)$ at later times.

Solve diffusion equation

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{\kappa} \frac{\partial T}{\partial t} = 0$$

$\kappa = \frac{K}{C_p}$

After long time we expect temperature to rise linearly with time.

Try $T_p = u(x) + \alpha t$ (Separation of Variables)
Thus, plugging equation

\[
\frac{\partial^2 U}{\partial x^2} - \frac{a}{K} = 0
\]

\[\Rightarrow U(x) = \frac{1}{2} \frac{a}{K} x^2 + ax + b\]

\[\text{(Ohms Law)}\]

\[\begin{align*}
\alpha &= -\frac{Q}{K} \\
-kU'(0) &= Q \\
U'(0) &= 0
\end{align*}\]

\[U(x) = \frac{1}{2} \frac{Q}{KD} (x - d)^2 \quad \alpha = \frac{Q}{Kd} = \frac{Q}{\text{cPd}}
\]

\[T_p = \frac{1}{2} \frac{Q}{KD} (x - d)^2 + \frac{Q}{\text{cPd}}
\]

These kind of solution that grow linearly with time are often used in Hamilton-Jacobi equations.

For the remainder of the class we will talk about how to use separation of variables to solve Helmholtz equation in spherical and cylindrical coordinates.
Let us now consider the Helmholtz equation in cylindrical coordinates:

\[
\nabla^2 \psi(p, \phi, z) + k^2 \psi(p, \phi, z) = 0
\]

The Laplacian in cylindrical coordinates becomes:

\[
\frac{1}{p} \frac{\partial}{\partial p} \left( p \frac{\partial \psi}{\partial p} \right) + \frac{1}{p^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0
\]

We can again use separation of variables:

\[
\psi(p, \phi, z) = R(p) \Phi(\phi) Z(z)
\]

(since orthonormal system)

Substituting:

\[
\frac{\Phi Z}{p} \frac{d}{dp} \left( p \frac{dR}{dp} \right) + \frac{RZ}{p^2} \frac{d^2 \Phi}{d\phi^2} + R \Phi \frac{dZ}{dz} + k^2 R \Phi Z = 0
\]

Divide by \( p \Phi Z \):

\[
\frac{1}{R} \frac{d}{dp} \left( p \frac{dR}{dp} \right) + \frac{1}{p^2 \Phi} \frac{d^2 \Phi}{d\phi^2} + k^2 = -\frac{1}{Z} \frac{d^2 Z}{dz^2}
\]

independent of \( Z \)

depends only on \( Z \)

So equals a constant \( = -k^2 \)
So the $z$ equations becomes

$$\frac{d^2 z}{dz^2} = c^2 z$$

and

$$\frac{1}{pR} \frac{d}{dp} \left( p \frac{dR}{dp} \right) + \frac{1}{p^2 \phi} \frac{d^2 \phi}{d\phi^2} = k^2 = -c^2$$

Now multiply by $p^2$.

$$(k^2 + \ell^2)$$

$$\frac{p}{R} \frac{d}{dp} \left( p \frac{dR}{dp} \right) + \ell^2 p^2 = -\frac{1}{\phi} \frac{d^2 \phi}{d\phi^2}$$

Depends on $p$

Depends on $\phi$

$$\frac{d^2 \Phi}{d\phi^2} = -m^2 \Phi$$

$\Phi = C \cos \Phi + D \sin \Phi$

The equation for $p$ then becomes

$$p \frac{d}{dp} \left( p \frac{dR}{dp} \right) + (n^2 p^2 - \ell^2) p = 0$$

Two cases

$n = 0$ \{ This is a straightforward equation \}

$n \neq 0$ \{ This is "Bessel's Differential Equation" \}
\[ \forall x = \frac{0}{x^2} = \frac{0}{x^2} \]

\[ 0 = 0 \]

\[ (x - 1)g + x^2 - x_2 = 0 \]

Look at term proportional to \( x^{-2} \).

\[ \sum_{j=0}^{\infty} \frac{x}{x^2 - 1} \]

So now plug this into the equation above.

\[ R(x) = f(x) \int \sum_{j=0}^{\infty} \frac{x}{x^2 - 1} \]

Assume power series solutions.

General properties of solutions to this are Bessel functions and we rewrite as so we

\[ 0 = \frac{d^2 x}{dp} + \left( \frac{x}{p} - 1 \right) \frac{ dp}{dp} \]

Useful to change parameters to \( x = p \).

\[ \frac{d^2 x}{dp} + \left( 1 - p^2 \right) \frac{ dp}{dp} \]

Now consider general case.

\[ R(x) = \left\{ \begin{array}{ll} 0 & x \neq 0 \\ \text{Laplace's equation} & x = 0 \end{array} \right. \]

\[ p \text{ is derivative} \]

\[ \text{Laplace's equation} \]

\[ \text{Laplace's equation} \]

\[ \text{Laplace's equation} \]

\[ \text{Laplace's equation} \]

\[ \text{Laplace's equation} \]

\[ \text{Laplace's equation} \]

\[ \text{Laplace's equation} \]
Now look at terms proportional to $X^\alpha-1$

$$(\alpha+1)X a_1 + \alpha a_1 = Y^2 a_1 = 0$$

Since $\alpha = \pm \gamma$

$$(\pm \gamma + 1)X \pm \gamma a_1 - Y^2 a_1 = 0$$

$$[Y^2 \pm \gamma - Y^2]a_1 = 0$$

$$\implies \pm \gamma a_1 = 0$$

$$\implies a_1 = 0$$

For $X^{j+\alpha-2}$ term for $j \geq 2$. Notice relate

$$(j+1)(j+\alpha-1)a_j + (j+\alpha)a_j + a_{j-2} - Y^2 a_j = 0$$

$$(j+\alpha)^2 a_j - Y^2 a_j = -a_{j-2}$$

$$\implies \left(\frac{j^2 + \alpha^2}{2}\right) + X^2 - X^2 a_j = -a_{j-2}$$

$$a_j = -\frac{a_{j-2}}{j(j+\alpha)}$$

So we have recursion for $a_j$ in terms of $a_{j-2}$. Since $a_1 = 0$

Only odd terms are kept so conventional to rewrite

$$a_{2j} = -\frac{1}{4j(j+\alpha)} a_{2j-2}$$

$$\sigma_{2j} = \frac{(-1)^j \Gamma(\alpha+1)}{2^{2j} j! \Gamma(j+\alpha+1)} a_0$$

$$\implies \bar{V}(X) = \left(\frac{X}{2}\right)^\gamma \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j+\gamma+1)} \left(\frac{X}{2}\right)^j$$

$$\bar{V}^{-\gamma} = \left(\frac{X}{2}\right)^\gamma \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j+\gamma+1)} \left(\frac{X}{2}\right)^j$$
These solutions are called Bessel functions of the first kind. If \( \nu \) is not an integer, these are independent.

\[
\begin{align*}
J_0(x) & \quad J_1(x) \\
J_1(x) & \quad J_2(x)
\end{align*}
\]

- Polynomial functions for small \( x \)
- Oscillatory functions for large \( x \)
- Crossover around \( x \approx \nu \)
- Thus have infinite number of roots

For \( \nu \) an integer, see \( J_{-\nu}(x) = (-1)^\nu J_\nu(x) \) so not independent!

For integers, replace \( J_{\pm \nu}(x) \) by \( J_\nu(x) \) and \( N_\nu(x) \) [Neumann function]

\[
N_\nu(x) = \frac{J_\nu(x) \cos \nu \pi - J_{-\nu}(x)}{\sin \nu \pi}
\]

For \( \nu \notin \mathbb{Z} \) clearly \( J_\nu(x) \) and \( N_\nu(x) \) are independent. \( \sqrt{N_\nu(x)} \) by L'Hopital rule

Also form Hankel Functions

\[
H^{(1)}_\nu(x) = J_\nu(x) + iN_\nu(x)
\]

\[
H^{(2)}_\nu(x) = J_\nu(x) - iN_\nu(x)
\]

[Important! Blow up at zero!]

\[
\lim_{x \to 0} N_\nu(x) \quad \lim_{x \to \infty} N_\nu(x)
\]
Using this equation, we can solve Laplace's equation.

Consider case if we had taken the case $k^2 = 0$ and
\[
\frac{d^2 Z}{dz^2} - \frac{1}{z^2} Z = -n^2 Z \quad (n^2 = k^2 + l^2 = l^2)
\]

Then corresponding Bessel equation after transforming to
\[
x = \pi p \
\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} - \left(1 + \frac{v^2}{x^2}\right) R = 0
\]

The solutions to these are the modified Bessel functions
\[
I_v(x) = -J_v(x) \quad I_v \text{ blow up at large } x
\]
\[
K_v(x) = \frac{\pi}{2} \Gamma(v+1) H_v^{(1)}(x) \quad K_v \text{ blow up at small } x.
\]
Bessel Functions all satisfy
\[ \Omega_{\nu-1}(x) + \Omega_{\nu+1}(x) = \frac{2\nu}{x} \Omega_{\nu}(x) \]
\[ \Omega_{\nu-1}(x) - \Omega_{\nu+1}(x) = 2 \frac{d\Omega_{\nu}(x)}{dx} \]

An important property that Bessel Functions Satisfy is that they are "orthogonal" in some sense (This is the basis of Sturm–Liouville theory) which we will re-visit in a few weeks.

Since Bessel functions eventually oscillate, they have infinite number of roots
\[ J_{\nu}(x_{\nu n}) = 0 \quad (n = 1, 2, 3, \ldots) \]
So now we show \[ \sqrt[\nu]{J_{\nu}(x_{\nu n} \rho/a)} \] for fixed \( \nu \geq 0 \), \( n = 1, 2, \ldots \) for orthogonal set on \( 0 \leq \rho \leq a \).

Namely, they satisfy
\[ a \int_{0}^{a} \rho J_{\nu}(x_{\nu n} \rho/a) J_{\nu}(x_{\nu n} \rho/a) \, d\rho = \frac{a^2}{2} \left[ J_{\nu+1}(x_{\nu n}) \right]_{0}^{a} \]
(Proof see Jackson Section 3.7 or Arfken)

So any function of \( \rho \) on interval \( 0 \leq \rho \leq a \) can be expanded in Fourier-Bessel series
\[ f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_{\nu}(x_{\nu n} \rho/a) \]

\[ A_{\nu n} = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})} \int_{0}^{a} \rho f(\rho) J_{\nu}(x_{\nu n} \rho/a) \, d\rho \]

This kind of series is convenient for function that vanish at \( \rho = a \) (Homogenous Dirichlet B.C. on a cylinder)
Consider

\[ \Phi = V(r, \phi) \]

\[ \nabla^2 \Phi = 0, \]

Since we have symmetry in \( \phi \) we use a separation of variables as before

\[ Z(r) = Ae^{-\frac{r}{L}} + Be^{\frac{r}{L}} \]

\[ \Phi(r, \phi) = C \cos m\phi + D \sin m\phi \]  

\((y = m \text{ integer since must be periodic})\)

\[ R(r) = E J_m (kr) + F N_m (kr) \]

Now look at B.C. We know \( \Phi = 0 \) at \( z = 0 \).

This implies \( A = -B \) so

\[ Z(z) = \sinh lz \]

Furthermore, we know that \( N_m(kr) \) blow up at \( r = 0 \) where potential is finite so \( F = 0 \).

\[ R(r) = E J_m (kr), \text{ Furthermore, } \Phi (r = 0) = 0 \text{ so } k = \frac{x_m}{\alpha} = k_m \]

So we know that

\[ \Phi(r, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m (k_{mn}r) \sinh(k_{mn}z) \]

\[ (A_{mn} \sin m\phi + B_{mn} \cos m\phi) \]
At \( z = L \) we have

\[
V(p, \varphi) = \sum_{m,n} \sinh (k_{mn}L) J_m(k_{mn}p) (A_{mn} \sin m\varphi + B_{mn} \cos m\varphi)
\]

This Fourier series in \( \varphi \) and Fourier-Bessel in \( p \).

So

\[
A_{mn} = \frac{2 \cosh (k_{mn}L)}{\pi a^2 J_{m+1}(k_{mn}a)} \int_0^{2\pi} \int_0^{2\pi} V(p, \varphi) J_m(k_{mn}p) \sin m\varphi \, dp \, d\varphi
\]

\[
B_{mn} = \frac{2 \cosh (k_{mn}L)}{\pi a^2 J_{m+1}(k_{mn}a)} \int_0^{2\pi} \int_0^{2\pi} V(p, \varphi) J_m(k_{mn}p) \cos m\varphi \, dp \, d\varphi
\]

Where we have used the Fourier-Bessel expansion formula

\[
V(p, \varphi) = \sum_{m,n} \left[ \frac{\int_0^{2\pi} \int_0^{2\pi} V(p, \varphi) J_m(k_{mn}p) \sin m\varphi \, dp \, d\varphi}{a^2 J_{m+1}(k_{mn}a)} \right] J_m(k_{mn}p) \sin m\varphi
\]

and

\[
f(\varphi) = \frac{1}{2} A_0 + \sum_{m=1}^{\infty} A_m \cos \left( \frac{2\pi m \varphi}{a} \right) + B_m \sin m\varphi
\]

\[
A_m = \frac{1}{\pi} \int_0^{2\pi} f(\varphi) \cos m\varphi \, d\varphi
\]

\[
B_m = \frac{1}{\pi} \int_0^{2\pi} f(\varphi) \sin m\varphi \, d\varphi
\]