Topic 35: Green's Functions II – Potential between Two Spherical Shells

In this topic we develop the tool to be used for finding the potential for an arbitrary charge distribution between two concentric spherical surfaces of radii a and b (a < b) having arbitrary potentials as boundary conditions. From our previous discussion of uniqueness theorems, we know that the solution to such a problem is unique. However, it is in general difficult to obtain the solution unless one can find the appropriate Green's Function. That is our primary goal in this topic. Much of the procedure employed here mirrors that used in Exercise 34-2, which involved the expansion of the Green's Function for all space in terms of Legendre polynomials. The approach used in this topic will lead to the expansion of the Green's Function in the space between the concentric spheres in terms of spherical harmonics.

From Topic 33 we know that if:

$$\nabla'^2 G_D(\vec{r}, \vec{r}') = -4\pi\delta(\vec{r}' - \vec{r}), \tag{33-6}$$

and

$$G_D(\vec{r}, \vec{r}') = 0 \text{ on surface S}, \tag{33-7}$$

then the potential in the volume V that is bounded by the surface S is:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{r}')G_D(\vec{r}, \vec{r}')d\tau' - \frac{1}{4\pi}\oint_S \varphi(\vec{r}')\frac{\partial G_D(\vec{r}, \vec{r}')}{\partial n'}da'. \tag{33-8}$$

There is no simple image solution for this problem, so we must employ a systematic procedure to find the Green's Function. Since the Green's Function is a potential for a specified charge distribution within a surface having specified potential, we know that once we have found a solution, it is the one and only solution by the uniqueness theorem.

Since the Green's Function is a solution to Laplace's equation for $\vec{r}' \neq \vec{r}$ we can express it in terms of spherical harmonics, ala equation (34-45), for $r' < r$ and $r' > r$:

$$G(\vec{r}, \vec{r}') = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(A_{nm}r'^n + \frac{B_{nm}}{r'^{n+1}}\right)Y_{nm}(\theta', \varphi') \; (r' < r), \tag{35-1}$$

$$G(\vec{r}, \vec{r}') = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(C_{nm}r'^n + \frac{D_{nm}}{r'^{n+1}}\right)Y_{nm}(\theta', \varphi') \; (r' > r). \tag{35-2}$$

G must vanish on the surfaces with radii a and b so we have that:

$$G(\vec{r},\vec{r}') = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm}\left(r'^n - \frac{a^{2n+1}}{r'^{n+1}}\right)Y_{nm}(\theta',\varphi') \;\; (r' < r), \tag{35-3}$$

and

$$G(\vec{r},\vec{r}') = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} D_{nm}\left(\frac{1}{r'^{n+1}} - \frac{r'^n}{b^{2n+1}}\right)Y_{nm}(\theta',\varphi') \;\; (r' > r), \tag{35-4}$$

Since G is a potential, it must be continuous across the sphere of radius r:

$$D_{nm}\left(\frac{1}{r^{n+1}} - \frac{r^n}{b^{2n+1}}\right) = A_{nm}\left(r^n - \frac{a^{2n+1}}{r^{n+1}}\right). \tag{35-5}$$

We can satisfy this requirement, and also satisfy the requirement that G be symmetric in $\vec{r}$ and $\vec{r}'$ by setting:

$$D_{nm} = f_{nm}(\theta,\varphi)\left(r^n - \frac{a^{2n+1}}{r^{n+1}}\right). \tag{35-6}$$

and

$$A_{nm} = f_{nm}(\theta,\varphi)\left(\frac{1}{r^{n+1}} - \frac{r^n}{b^{2n+1}}\right). \tag{35-7}$$

We can now combine equations (35-3) and (35-4) into the following one:

$$G(\vec{r},\vec{r}') = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_{nm}(\theta,\varphi)\left(r_<^n - \frac{a^{2n+1}}{r_<^{n+1}}\right)\left(\frac{1}{r_>^{n+1}} - \frac{r_>^n}{b^{2n+1}}\right)Y_{nm}(\theta',\varphi'). \tag{35-8}$$

where $r_<$ is the smaller of r and r' and $r_>$ is the larger of r and r'. To determine f_{nm} we impose the condition that the electric field must be continuous across the sphere of radius r (except at $\vec{r}$):

$$-\left.\frac{\partial G}{\partial r'}\right|_+ = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_{nm}(\theta,\varphi)\left(r^n - \frac{a^{2n+1}}{r^{n+1}}\right)\left(\frac{n+1}{r^{n+2}} + \frac{nr^{n-1}}{b^{2n+1}}\right)Y_{nm}(\theta',\varphi'), \tag{35-9}$$

and

$$-\left.\frac{\partial G}{\partial r'}\right|_- = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_{nm}(\theta,\varphi)\left(-nr^{n-1} - \frac{(n+1)a^{2n+1}}{r^{n+2}}\right)\left(\frac{1}{r^{n+1}} - \frac{r^n}{b^{2n+1}}\right)Y_{nm}(\theta',\varphi'). \tag{35-10}$$

So,

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)f_{nm}(\theta,\varphi)\left(\frac{a^{2n+1}}{b^{2n+1}}-1\right)Y_{nm}(\theta',\varphi')=0, \quad \text{if } (\theta'\neq\theta,\varphi'\neq\varphi). \qquad (35\text{-}11)$$

It is easy to identify f_{nm} by using the completeness relation for spherical harmonics:

$$f_{nm}(\theta,\varphi)=K\frac{1}{(2n+1)}\frac{1}{\left(1-\frac{a^{2n+1}}{b^{2n+1}}\right)}Y^*_{nm}(\theta,\varphi), \qquad (35\text{-}12)$$

where K is a constant. Note that this choice for f_{nm} also completes the requirement that G is symmetric in $\vec{r}$ and $\vec{r}\,'$, and provides the needed singularity at the point $\vec{r}\,'=\vec{r}$. So we now have that:

$$G(\vec{r},\vec{r}\,')=K\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{1}{(2n+1)}\frac{1}{\left(1-\frac{a^{2n+1}}{b^{2n+1}}\right)}\left(r_<^n-\frac{a^{2n+1}}{r_<^{n+1}}\right)\left(\frac{1}{r_>^{n+1}}-\frac{r_>^n}{b^{2n+1}}\right)Y^*_{nm}(\theta,\varphi)Y_{nm}(\theta',\varphi').$$

We determine K by using Gauss' Law and by requiring the charge between spheres with radii slightly larger and slightly smaller than r be equal to $4\pi\varepsilon_0$. We integrate (35-9) to find the charge enclosed by a sphere with radius slightly larger than r:

$$Q_+=-\varepsilon_0\oint\frac{\partial G}{\partial r'}\bigg|_+ da'=\varepsilon_0 f_{00}(\theta,\varphi)\left(1-\frac{a}{r}\right)\sqrt{4\pi},$$

where we have used the orthogonality of the spherical harmonics. Similarly we find the charge within a sphere having radius slightly smaller than r:

$$Q_-=-\varepsilon_0\oint\frac{\partial G}{\partial r'}\bigg|_- da'=\varepsilon_0 f_{00}(\theta,\varphi)(-a)\left(\frac{1}{r}-\frac{1}{b}\right)\sqrt{4\pi}.$$

The total charge between these two spheres must be $4\pi\varepsilon_0$:

$$Q_+-Q_-=\varepsilon_0 f_{00}(\theta,\varphi)\left[\left(1-\frac{a}{r}\right)+\left(\frac{a}{r}-\frac{a}{b}\right)\right]\sqrt{4\pi}=K\varepsilon_0=4\pi\varepsilon_0.$$

So K = 4π and we have for the Green's Function:

$$G(\vec{r},\vec{r}') = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{4\pi\left(r_<^n - \dfrac{a^{2n+1}}{r_<^{n+1}}\right)\left(\dfrac{1}{r_>^{n+1}} - \dfrac{r_>^n}{b^{2n+1}}\right)}{(2n+1)\left(1-\dfrac{a^{2n+1}}{b^{2n+1}}\right)} Y^*_{nm}(\theta,\varphi)Y_{nm}(\theta',\varphi'). \qquad (35\text{-}13)$$

If we let a go to zero and b go to infinity, we will get the Green's Function for all space, $1/|\vec{r}'-\vec{r}|$:

$$\frac{1}{|\vec{r}'-\vec{r}|} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{4\pi}{(2n+1)}\frac{r_<^n}{r_>^{n+1}} Y^*_{nm}(\theta,\varphi)Y_{nm}(\theta',\varphi'). \qquad (35\text{-}14)$$

Comparing with equation (34-39) we obtain the *addition theorem* for spherical harmonics:

$$P_n(\cos\gamma) = \sum_{m=-n}^{n} \frac{4\pi}{(2n+1)} Y^*_{nm}(\theta,\varphi)Y_{nm}(\theta',\varphi'), \qquad (35\text{-}15)$$

where γ is the angle between $\vec{r}'$ and $\vec{r}$.

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Exercise 35-1

A ring of charge has uniform linear charge density. The ring has radius R and lies in the xy plane. The total charge of the bring is Q. The ring is between two concentric, grounded metal spheres of radii a and b (a < b). Find the potential between the spheres.

Solution 35-1

We use equations (33-8) and (35-13). The charge density is given by:

$$\rho(\vec{r}') = \frac{Q\delta(r'-R)\delta(\cos\theta')}{2\pi R^2}.$$

The potential is:

$$\Phi(\vec{r}) = \frac{Q}{4\pi R^2\varepsilon_0}\int_V \delta(r'-R)\sum_{n=0}^{\infty} \frac{4\pi\left(r_<^n - \dfrac{a^{2n+1}}{r_<^{n+1}}\right)\left(\dfrac{1}{r_>^{n+1}} - \dfrac{r_>^n}{b^{2n+1}}\right)}{(2n+1)\left(1-\dfrac{a^{2n+1}}{b^{2n+1}}\right)} Y^*_{n0}(\theta,\varphi)Y_{n0}\left(\frac{\pi}{2},\varphi'\right)r'^2 dr'$$

From the definition of the spherical harmonics and associated Legendre functions we have:

$$Y_{n0}(\theta,\varphi)=\sqrt{\frac{2n+1}{4\pi}}P_n^0(\cos\theta)$$

and

$$P_n^0(x)=P_n(x)$$

so

$$\Phi(\vec{r})=\frac{Q}{4\pi R^2\varepsilon_0}\sum_{n=0}^{\infty}\frac{P_n(\cos\theta)P_n(0)}{\left(1-\frac{a^{2n+1}}{b^{2n+1}}\right)}\int_V \delta(r'-R)\left(r_<^n-\frac{a^{2n+1}}{r_<^{n+1}}\right)\left(\frac{1}{r_>^{n+1}}-\frac{r_>^n}{b^{2n+1}}\right)r'^2dr'.$$

The integral can be written:

$$\int_V \delta(r'-R)\left(r_<^n-\frac{a^{2n+1}}{r_<^{n+1}}\right)\left(\frac{1}{r_>^{n+1}}-\frac{r_>^n}{b^{2n+1}}\right)r'^2dr'=\int_a^r \delta(r'-R)\left(r'^n-\frac{a^{2n+1}}{r'^{n+1}}\right)\left(\frac{1}{r^{n+1}}-\frac{r^n}{b^{2n+1}}\right)r'^2dr'+$$
$$\int_r^b \delta(r'-R)\left(r^n-\frac{a^{2n+1}}{r^{n+1}}\right)\left(\frac{1}{r'^{n+1}}-\frac{r'^n}{b^{2n+1}}\right)r'^2dr'=\left(r_<^n-\frac{a^{2n+1}}{r_<^{n+1}}\right)\left(\frac{1}{r_>^{n+1}}-\frac{r_>^n}{b^{2n+1}}\right)R^2$$

where the greater and lesser than signs now refer to r and R:

$$\Phi(\vec{r})=\frac{Q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{P_n(\cos\theta)P_n(0)}{\left(1-\frac{a^{2n+1}}{b^{2n+1}}\right)}\left(r_<^n-\frac{a^{2n+1}}{r_<^{n+1}}\right)\left(\frac{1}{r_>^{n+1}}-\frac{r_>^n}{b^{2n+1}}\right).$$

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