Topic 33: Green's Functions I – Solution to Poisson's Equation with Specified Boundary Conditions

This is the first of five topics that deal with the solution of electromagnetism problems through the use of *Green's functions*. We will begin with the presentation of a procedure that permits the solution of electrostatic problems with specified charge distributions within a volume V surrounded by a surface S that has specified boundary conditions for either the potential or the normal component of electric field. To set the stage, consider the application of Green's theorem for such a geometry:

$$\int_V \left(\psi(\vec{r}\,')\nabla'^2\varphi(\vec{r}\,') - \varphi(\vec{r}\,')\nabla'^2\psi(\vec{r}\,')\right)d\tau' = \oint_S \left(\psi(\vec{r}\,')\frac{\partial\varphi(\vec{r}\,')}{\partial n'} - \varphi(\vec{r}\,')\frac{\partial\psi(\vec{r}\,')}{\partial n'}\right)da' . \qquad (33\text{-}1)$$

Note that we have chosen a prime to denote the variable of integration. We choose the two functions as follows. Let $\varphi(\vec{r}\,')$ be the electrostatic potential in V. Therefore

$$\nabla'^2\varphi(\vec{r}\,') = -\frac{\rho(\vec{r}\,')}{\varepsilon_0}, \qquad (33\text{-}2)$$

where $\rho(\vec{r}\,')$ is the charge density. For the other function we choose:

$$\psi(\vec{r}\,') = \frac{1}{|\vec{r} - \vec{r}\,'|} \qquad (33\text{-}3)$$

where $\vec{r}$ is a fixed point in V. Equation (33-1) becomes:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{r}\,')}{|\vec{r}\,' - \vec{r}|}d\tau' + \frac{1}{4\pi}\oint_S \left(\frac{1}{|\vec{r}\,' - \vec{r}|}\frac{\partial\varphi(\vec{r}\,')}{\partial n'}\right)da' - \frac{1}{4\pi}\oint_S \left(\varphi(\vec{r}\,')\frac{\partial}{\partial n'}\frac{1}{|\vec{r}\,' - \vec{r}|}\right)da' . \qquad (33\text{-}4)$$

As S recedes to infinity the surface integrals vanish, and we recover Coulomb's Law, which is not particularly interesting or useful. Equation (33-4) does suggest the possibility of a useful expression for evaluating the potential at points in V if S corresponds to an actual physical boundary. However, as it stands, this will not work. This is due to the fact that one would need to specify both the potential and its derivative on the boundary in order to evaluate the integrals. But such a case would correspond to an over-specification since either type of boundary condition suffices to uniquely determine the potential inside V.

Dirichlet Green's Function

If we modify our choice for $\psi(\vec{r}\,')$ the situation improves. Suppose that instead of Equation (33-3) we use the following function:

$$\psi(\vec{r}\,') = G_D(\vec{r}, \vec{r}\,'), \qquad (33\text{-}5)$$

where the *Green's Function* G is subject to the conditions:

$$\nabla'^2 G_D(\vec{r}, \vec{r}\,') = -4\pi\delta(\vec{r}\,' - \vec{r}), \qquad (33\text{-}6)$$

and

$$G_D(\vec{r}, \vec{r}\,') = 0 \text{ on S.} \qquad (33\text{-}7)$$

Then Equation (33-4) becomes:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{r}\,')G_D(\vec{r}, \vec{r}\,')d\tau' - \frac{1}{4\pi}\oint_S \varphi(\vec{r}\,')\frac{\partial G_D(\vec{r}, \vec{r}\,')}{\partial n'}da'. \qquad (33\text{-}8)$$

Equation (33-8) can be used to solve Dirichlet type problems for which the potential is specified on the surface.

Neumann Green's Function

Suppose that instead of equation (33-5) we use the following function:

$$\psi(\vec{r}\,') = G_N(\vec{r}, \vec{r}\,'), \qquad (33\text{-}9)$$

where G is subject to the conditions:

$$\nabla'^2 G_N(\vec{r}, \vec{r}\,') = -4\pi\delta(\vec{r}\,' - \vec{r}), \qquad (33\text{-}10)$$

and

$$\frac{\partial G_N(\vec{r}, \vec{r}\,')}{\partial n'} = -\frac{4\pi}{S} \text{ on S.} \qquad (33\text{-}11)$$

We have used the symbol S in equation (33-11) to denote the surface area of the surface S. Then Equation (33-8) becomes:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{r}\,')G_N(\vec{r}, \vec{r}\,')d\tau' + \frac{1}{4\pi}\oint_S G_N(\vec{r}, \vec{r}\,')\frac{\partial \varphi(\vec{r}\,')}{\partial n'}da' + \frac{1}{S}\oint_S \varphi(\vec{r}\,')da'. \qquad (33\text{-}12)$$

Equation (33-12) can be used to solve Neumann type problems for which the normal derivative of the potential is specified on the surface. It was necessary to impose condition (33-11) on the Neumann Green's Function to be consistent with equation (33-10).

Symmetry Condition for Dirichlet Green's Function

Let $\psi(\vec{r}\,')=G_D(\vec{x},\vec{r}\,')$ and let $\varphi(\vec{r}\,')=G_D(\vec{y},\vec{r}\,')$ for a Dirichlet type Green's Function, where $\vec{x}$ and $\vec{y}$ are two points inside of V. Inserting these into Green's theorem we get:

$$\int_V \left(G_D(\vec{x},\vec{r}\,')\nabla'^2 G_D(\vec{y},\vec{r}\,')-G_D(\vec{y},\vec{r}\,')\nabla'^2 G_D(\vec{x},\vec{r}\,')\right)d\tau'=0$$

We have used the fact that the Dirichlet Green's Function vanishes on S. Now let's use condition (33-6) to obtain:

$$-4\pi G_D(\vec{x},\vec{y})+4\pi G_D(\vec{y},\vec{x})=0. \qquad (33\text{-}13)$$

So the Dirichlet Green's Function must be symmetric under exchange of its two position variables.

Dirichlet Green's Function for a Sphere

The search for a Dirichlet Green's Function is equivalent to the search for an image charge, a procedure that is covered in most undergraduate courses. In each case we need to find a function that is a potential within volume V. The potential is due to a point charge in V, added to a potential due to charges outside the region of interest. The potential is required to vanish on S for both the Green's Function problem and for the usual form of the image problem in which the bounding surface is a grounded metallic object.

As an example we state the result for such a search for a sphere of radius a. Suppose we are interested in the Green's Function for *interior problems* (i.e. we want to calculate the potential within the sphere). By considering the elementary image problem for a sphere (consult your undergraduate text), it is easy to find the appropriate Green's Function:

$$G_D(\vec{r},\vec{r}\,')=\frac{1}{\left|\vec{r}\,'-\vec{r}\right|}-\frac{(a/r)}{\left|\vec{r}\,'-\frac{a^2}{r^2}\vec{r}\right|}$$

which can be written as:

$$G_D(\vec{r},\vec{r}\,')=\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}}-\frac{1}{\sqrt{\frac{r^2r'^2}{a^2}+a^2-2rr'\cos\gamma}}. \qquad (33\text{-}14)$$

In this equation, γ is the angle between the vectors $\vec{r}$ and $\vec{r}'$. The Green's function for the exterior problem is easily seen to be exactly the same (except that $\vec{r}$ and $\vec{r}'$ are outside the sphere in this case).

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Exercise 33-1

Consider an isolated metal sphere of radius a whose center is located at the origin. The sphere has no charge on it. A positive charge Q is located at z = –R, and a charge –Q is located at z = R. Let's find the potential outside the sphere.

Solution 33-1

The potential on the sphere is zero, which can be seen by calculating the work done in moving a charge from infinity to the surface of the sphere along a line in the xy plane. Equation (33-8) becomes:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{r}')G_D(\vec{r},\vec{r}')d\tau', \qquad (33\text{-}15)$$

where V is the volume exterior to the sphere. This equation yields the potential at all points outside the sphere. By using the specified charge distribution, we get:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \left(Q\delta\left(\vec{r}' + R\hat{k}\right) - Q\delta\left(\vec{r}' - R\hat{k}\right)\right)G_D(\vec{r},\vec{r}')d\tau'. \qquad (33\text{-}16)$$

So we have:

$$\varphi(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\left[G_D\left(\vec{r},-R\hat{k}\right) - G_D\left(\vec{r},R\hat{k}\right)\right]. \qquad (33\text{-}17)$$

Using equation (33-14):

$$G_D\left(\vec{r},R\hat{k}\right) = \frac{1}{R\sqrt{1 - \frac{2r}{R}\cos\theta + \frac{r^2}{R^2}}} - \frac{(a/r)}{R\sqrt{1 - \frac{2a^2}{rR}\cos\theta + \frac{a^4}{R^2r^2}}}, \qquad (33\text{-}18)$$

where θ is the usual polar angle. With equation (33-18) is easy to see that:

$$G_D\left(\vec{r},-R\hat{k}\right) = \frac{1}{R\sqrt{1 + \frac{2r}{R}\cos\theta + \frac{r^2}{R^2}}} - \frac{(a/r)}{R\sqrt{1 + \frac{2a^2}{rR}\cos\theta + \frac{a^4}{R^2r^2}}}. \qquad (33\text{-}19)$$

The answer is thus specified in terms of equations (33-17), (33-18), and (33-19).

An interesting check of the results can be obtained by letting R approach infinity. In this case:

$$G_D\left(\vec{r}, R\hat{k}\right) = \frac{1}{R}\left(1 + \frac{r}{R}\cos\theta\right) - \frac{a}{rR}\left(1 + \frac{a^2}{rR}\cos\theta\right)$$

and

$$G_D\left(\vec{r}, -R\hat{k}\right) = \frac{1}{R}\left(1 - \frac{r}{R}\cos\theta\right) - \frac{a}{rR}\left(1 - \frac{a^2}{rR}\cos\theta\right).$$

Combining terms we have:

$$\varphi(\vec{r}) = -E_0 r\cos\theta + \frac{E_0 a^3 \cos\theta}{r^2}, \tag{33-20}$$

where $E_0 = \dfrac{2Q}{4\pi\varepsilon_0 R^2}$ is the uniform field produced by the external charges. This is the familiar expression for the electric field outside of an uncharged conduction sphere in a uniform field.

The charge density on the sphere is easily found to be:

$$\sigma = 3\varepsilon_0 E_0 \cos\theta. \tag{33-21}$$