NINE

EIGENFUNCTIONS, EIGENVALUES, AND GREEN'S FUNCTIONS

Eigenvalue problems and some general properties of the eigenfunctions of Hermitian differential operators are discussed in the first two sections of this chapter. The solution of inhomogeneous problems is treated in Section 9–4, with emphasis on the use of Green's functions for describing particular solutions. Various methods for finding Green's functions are presented, and in Section 9–5 some examples from electrodynamics are given.

9–1 SIMPLE EXAMPLES OF EIGENVALUE PROBLEMS

We have already discussed (in Chapter 6) eigenvalues of matrices. These were values of the parameter λ for which nontrivial solutions v of the equation

$$Mv = \lambda v \tag{9-1}$$

exist, where M is the matrix under consideration. We shall now generalize this idea and consider the eigenvalue problem for *any* linear operator M, whether it be a matrix, differential operator, or integral operator. The basic equation is just the one written above, with v being whatever kind of object M can operate on. If M is a matrix, v is a column vector; if M is a differential operator, v is a function, and so forth. Solutions v of the above equation are called *eigenfunctions, eigenvectors*, and so on.

Eigenvalue problems have a great deal in common whether we are dealing with matrices or more general linear operators. The most important such problems in physics involve three-dimensional second-order partial differential operators. The examples and notation of this chapter will reflect the importance of such problems, but the student should not forget that most of our techniques may be generalized to any other type of eigenvalue problem.

As a very simple example, consider the problem of finding the eigenvalues and eigenvectors of the matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{9-2}$$

The equation

$$Mv = \lambda v$$

is equivalent to the two equations

$$(1 - \lambda)v_1 + v_2 = 0$$
$$v_1 + (1 - \lambda)v_2 = 0$$

if we set

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The eigenvalues and associated eigenvectors are

$$\lambda_1 = 0 \quad v^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \lambda_2 = 2 \quad v^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{9-3}$$

We may note several facts.

1. Because the original problem is homogeneous, eigensolutions are only determined up to an arbitrary constant multiplicative factor.

2. If λ equals an eigenvalue, the corresponding inhomogeneous problem

$$(M - \lambda)v = u \qquad (u \neq 0) \tag{9-4}$$

generally has no solution. In fact, a solution will exist only if u satisfies certain conditions; with the matrix M of (9-2), for example, and $\lambda = 0$, u_1 must equal u_2 in order that (9-4) have a solution. In other words, u must be "orthogonal" to the eigenvector $v^{(1)}$ belonging to the eigenvalue $\lambda = 0$; this trivial observation may be generalized [compare Eq. (9-29)].

3. The "dot product" of our two eigenvectors is zero.

4. An arbitrary 2-component vector can be written as a linear combination of the eigenvectors.

These four properties are quite general and will occur over and over as we encounter other eigenvalue problems.

EXAMPLE

Consider the equation for a string, fastened at $x = 0$ and $x = L$, vibrating with angular frequency ω (one-dimensional Helmholtz equation).

$$\frac{d^2u}{dx^2} + k^2u = 0 \qquad \left(k = \frac{\omega}{c}\right)$$

Since $u = 0$ when $x = 0$, $u = a \sin kx$. Since $u = 0$ when $x = L$, $k = n\pi/L$, where $n = 1, 2, 3, \ldots$. These are the eigenvalues k. Technically if we, consider d^2/dx^2 to be the linear operator, then we should call $-k^2$ the eigenvalue in this problem. This distinction is not very important, however, and we shall consider $-k^2$, or k^2, or k to be entitled to the name eigenvalue.

Note that a differential equation needs both a region of interest and boundary conditions in order to define an eigenvalue problem, and that the problem must be homogeneous. The student is invited to discover the analogies, in this second example, of the four properties mentioned after the first example.

Finally, consider a three-dimensional example.

EXAMPLE

$$\nabla^2u + k^2u = 0 \text{ inside the sphere } r = R \tag{9-5}$$

with the boundary condition: $u = 0$ on $r = R$.

Solutions are

$$u = j_l(kr)P_l^m(\cos\theta)e^{\pm im\phi} \qquad m, l \text{ integers}, l \geqslant m$$

Eigenvalues are fixed by the condition

$$j_l(kR) = 0$$

Thus the eigenvalues are

$$k \approx \frac{3.14}{R}, \frac{4.49}{R}, \ldots$$

9–2 GENERAL DISCUSSION

Consider a linear differential operator L and the eigenvalue problem

$$Lu(\mathbf{x}) = \lambda u(\mathbf{x})$$

We suppose that a region Ω has been specified, and suitable boundary conditions imposed.

L is said to be *Hermitian* if

$$\int_\Omega u^*(\mathbf{x}) Lv(\mathbf{x})\, d^3x = \left[\int_\Omega v^*(\mathbf{x}) Lu(\mathbf{x})\, d^3x\right]^*$$

where the asterisk denotes complex conjugation and u and v are arbitrary functions *obeying the boundary conditions*. The quantity on the left side of the above equation is often called the "u, v matrix element of L," or the "matrix element of L between u and v," or just L_{uv}. The connection with Hermitian matrices, as defined in Chapter 6, should be obvious.

Suppose L is Hermitian. Consider a particular eigenvalue λ_i and an eigenfunction u_i belonging to λ_i. Then

$$Lu_i(\mathbf{x}) = \lambda_i u_i(\mathbf{x})$$

The corresponding equation for the pair λ_j, $u_j(\mathbf{x})$ is

$$Lu_j(\mathbf{x}) = \lambda_j u_j(\mathbf{x})$$

Then

$$\int_\Omega u_j^*(\mathbf{x}) Lu_i(\mathbf{x})\, d^3x = \lambda_i \int_\Omega u_j^*(\mathbf{x}) u_i(\mathbf{x})\, d^3x$$

$$\int_\Omega u_i^*(\mathbf{x}) Lu_j(\mathbf{x})\, d^3x = \lambda_j \int_\Omega u_i^*(\mathbf{x}) u_j(\mathbf{x})\, d^3x$$

Since L is Hermitian, the left sides are complex conjugates of each other. Therefore,

$$(\lambda_i - \lambda_j^*) \int_\Omega u_j^*(\mathbf{x}) u_i(\mathbf{x})\, d^3x = 0 \tag{9-6}$$

At this point it should have become clear that we are just repeating the steps of the proof in Section 6–5 of certain properties of the eigenvalues and eigenvectors of a Hermitian matrix. We draw the same conclusions from the relation (9-6) by considering the two cases $i = j$ and $\lambda_i \neq \lambda_j$. Namely:

1. The eigenvalues of a Hermitian differential operator are real.
2. Eigenfunctions of a Hermitian differential operator, belonging to different eigenvalues, are orthogonal.

By $u(\mathbf{x})$ and $v(\mathbf{x})$ being orthogonal, we mean [compare Eq. (6-94)]

$$u \cdot v \equiv \int_{\Omega} u^*(\mathbf{x}) v(\mathbf{x}) \, d^3x = 0 \tag{9-7}$$

The most familiar set of orthogonal functions is the set of trigonometric functions. They are the eigenfunctions associated with the eigenvalue problem

$$\begin{gathered} \frac{d^2u}{dx^2} + \lambda u = 0 \qquad 0 \leqslant x \leqslant 2\pi \\ u(0) = u(2\pi) \qquad u'(0) = u'(2\pi) \end{gathered} \tag{9-8}$$

Is the operator d^2/dx^2 on the interval $0 \leqslant x \leqslant 2\pi$ with the periodic boundary conditions of (9-8) Hermitian? Let us see.

$$\int_0^{2\pi} u^* \frac{d^2}{dx^2} v \, dx = u^* \frac{dv}{dx}\bigg|_0^{2\pi} - \int_0^{2\pi} \frac{du^*}{dx} \frac{dv}{dx} \, dx$$

The integrated part vanishes because of the periodicity of u and v. Integrating once more by parts,

$$\int_0^{2\pi} u^* \frac{d^2}{dx^2} v \, dx = -\frac{du^*}{dx} v \bigg|_0^{2\pi} + \int_0^{2\pi} v \frac{d^2u^*}{dx^2} \, dx$$

Again the integrated part vanishes, so that

$$\int_0^{2\pi} u^* \frac{d^2}{dx^2} v \, dx = \left(\int_0^{2\pi} v^* \frac{d^2}{dx^2} u \, dx \right)^*$$

and d^2/dx^2 is indeed Hermitian, when the periodic boundary conditions of (9-8) are specified.

As another illustration of an eigenvalue problem involving a Hermitian operator, consider the so-called *Sturm–Liouville differential equation*

$$\frac{d}{dx}\left[p(x) \frac{du(x)}{dx}\right] - q(x)u(x) + \lambda\rho(x)u(x) = 0 \tag{9-9}$$

$p(x)$, $q(x)$, and $\rho(x)$ are real functions, and in addition $\rho(x)$ is assumed to be nonnegative on the interval in question. The function $u(x)$ will be required to vanish at both ends of the interval.[1] We must consider two points.

1. Is the operator $L = p(d^2/dx^2) + p'(d/dx) - q$ Hermitian? The verification proceeds just as for the simpler case of d^2/dx^2; we shall leave it for the student.

[1] Other boundary conditions could be specified; for example, $u(x)$ could vanish at one end and its derivative at the other. The student is invited to construct the most general boundary condition for which the succeeding arguments remain valid.

2. What is the effect of $\rho(x)$ not being 1? It is straightforward to repeat all the preceding arguments with a so-called *density function* $\rho(x)$ present in the term containing the eigenvalue. Orthogonality now means

$$u_i \cdot u_j \equiv \int_a^b u_i^*(x)u_j(x)\rho(x)\,dx = 0 \qquad \text{if} \qquad \lambda_i \neq \lambda_j \tag{9-10}$$

Often two or more eigenfunctions belong to the same eigenvalue. This situation is referred to as *degeneracy*, and the eigenvalue in question is said to be degenerate. (Compare the corresponding discussion for matrices on p. 152.) Note that an arbitrary linear combination of eigenfunctions belonging to a degenerate set is again an eigenfunction with the same eigenvalue. By using the Gram–Schmidt procedure (see p. 152) we can always construct an orthogonal set of eigenfunctions.

Thus it is possible to arrange things so that the eigenfunctions of a Hermitian operator form an *orthonormal* set, being *orthogonal*

$$u_i \cdot u_j = 0 \qquad (i \neq j)$$

and *normalized*

$$u_i \cdot u_i = 1$$

These two conditions may be written

$$u_i \cdot u_j = \left[\int d^3x\, u_i^*(\mathbf{x})u_j(\mathbf{x})\rho(\mathbf{x})\right] = \delta_{ij} \tag{9-11}$$

Property 4 noted at the beginning of this chapter, together with the example of Fourier series, suggest that it is possible to expand any function, obeying the appropriate conditions, in a series of eigenfunctions. That is, the eigenfunctions of a Hermitian operator form a *complete set* under very general conditions. This is just the infinite dimensional generalization of the theorem we have already proved in Chapter 6, p. 158. We shall not prove this property here but it is in fact true for all the commonly encountered differential equations in physics. The student interested in a mathematical treatment of the subject should consult a reference such as Courant and Hilbert (C10) Volume I, Chapters II and IV, or Titchmarsh (T2).

Let us therefore consider an expansion of the form

$$f(\mathbf{x}) = \sum_n c_n u_n(\mathbf{x}) \tag{9-12}$$

If we have chosen the u_n to be orthonormal, we can determine the c_n very easily:

$$u_m \cdot f = \sum_n c_n u_m \cdot u_n = \sum_n c_n \delta_{mn} = c_m \tag{9-13}$$

A useful formal relation follows by substituting this result back into the original series:

$$
\begin{aligned}
f(\mathbf{x}) &= \sum_n u_n(\mathbf{x}) u_n \cdot f \\
&= \sum_n u_n(\mathbf{x}) \int_\Omega d^3x' u_n^*(\mathbf{x}') f(\mathbf{x}') \rho(\mathbf{x}') \\
&= \int_\Omega d^3x' f(\mathbf{x}') \rho(\mathbf{x}') \sum_n u_n(\mathbf{x}) u_n^*(\mathbf{x}')
\end{aligned}
$$

Therefore,

$$\rho(\mathbf{x}') \sum_n u_n(\mathbf{x}) u_n^*(\mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \tag{9-14}$$

This is sometimes called the *completeness relation* and sometimes the *closure* property of the eigenfunctions. Note the interesting comparison with the orthogonality relation (9-11).

9–3 SOLUTIONS OF BOUNDARY-VALUE PROBLEMS AS EIGENFUNCTION EXPANSIONS

There is a close relationship between the expansion of a function in terms of eigenfunctions of a differential operator and the solution of a partial differential equation obtained by either the method of separation of variables or the method of integral transforms.

Suppose a partial differential equation contains a differential operator L with respect to one of the variables. If the solution is expressed as a sum over the eigenfunctions of L, then the combination of derivatives L will be eliminated from the equation and replaced by the eigenvalues, thus reducing the equation to one with fewer variables.

EXAMPLE

Consider the drum-head problem treated in Chapter 8, beginning with Eq. (8-46).

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{9-15}$$

We could eliminate the derivatives with respect to r, θ, or t first; choose θ. The operator $L_\theta = d^2/d\theta^2$ has eigenfunctions $e^{\pm in\theta}$ belonging to eigenvalues $-n^2$, where n is an integer for the boundary conditions $u(2\pi) = u(0)$,

$u'(2\pi) = u'(0)$. Thus we express u as the sum

$$u = \sum_{n=0}^{\infty} f_n(r, t)(e^{in\theta} + a_n e^{-in\theta}) \tag{9-16}$$

where the coefficients f_n are functions of the remaining variables r and t. The differential equation is reduced to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_n}{\partial r}\right) - \frac{n^2}{r^2}f_n - \frac{1}{c^2}\frac{\partial^2 f_n}{\partial t^2} = 0 \tag{9-17}$$

Next, eliminate the r derivatives by expanding f_n in eigenfunctions of the operator

$$L_r = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2} \tag{9-18}$$

with the boundary conditions: $u(r = 0)$ is finite and $u(r = R) = 0$. These are the Bessel functions $J_n(k_{jn} r)$ with eigenvalues $-k_{jn}^2$ such that $J_n(k_{jn} R) = 0$. Thus

$$f_n(r, t) = \sum_{j=1}^{\infty} g_{jn}(t) J_n(k_{jn} r) \tag{9-19}$$

where the coefficients g_{jn} depend on the remaining variable t. The differential equation is reduced to

$$k_{jn}^2 g_{jn}(t) + \frac{1}{c^2}\frac{d^2 g_{jn}}{dt^2} = 0 \tag{9-20}$$

with solutions

$$g_{jn}(t) = e^{\pm i\omega_{jn} t} \qquad \omega_{jn} = ck_{jn} \tag{9-21}$$

Collecting results, we have the solution (8-54) obtained by separation of variables

$$u(r, \theta, t) = \sum_{j,n=1}^{\infty} A_{jn} J_n(k_{jn} r)(\sin n\theta + B_n \cos n\theta)(\sin \omega_{jn} t + C_{jn} \cos \omega_{jn} t) \tag{9-22}$$

9–4 INHOMOGENEOUS PROBLEMS. GREEN'S FUNCTIONS

As pointed out in Chapter 8, a problem may be inhomogeneous because of the differential equation or because of the boundary conditions. Usually a transformation from one form of inhomogeneity to the other is possible, as we shall see near the end of this section.

First, let us consider the inhomogeneous equation

$$Lu(\mathbf{x}) - \lambda u(\mathbf{x}) = f(\mathbf{x}) \tag{9-23}$$

over a domain Ω, with L a Hermitian differential operator, with $u(\mathbf{x})$ subject to the usual type of (homogeneous) boundary conditions, and λ a given constant. To solve the equation, expand $u(\mathbf{x})$ and $f(\mathbf{x})$ in eigenfunctions of the operator L.

$$u(\mathbf{x}) = \sum_n c_n u_n(\mathbf{x}) \qquad f(\mathbf{x}) = \sum_n d_n u_n(\mathbf{x}) \tag{9-24}$$

Equation (9-23) becomes

$$\sum_n c_n(\lambda_n - \lambda)u_n(\mathbf{x}) = \sum_n d_n u_n(\mathbf{x})$$

Therefore, since the eigenfunctions $u_n(\mathbf{x})$ are linearly independent,

$$c_n = \frac{d_n}{\lambda_n - \lambda}$$

or, since

$$d_n = u_n \cdot f$$

we may write this as

$$c_n = \frac{u_n \cdot f}{\lambda_n - \lambda}$$

Therefore,

$$\begin{aligned} u(\mathbf{x}) &= \sum_n \frac{u_n u_n \cdot f}{\lambda_n - \lambda} \\ &= \sum_n \frac{u_n(\mathbf{x})}{\lambda_n - \lambda} \int_\Omega u_n^*(\mathbf{x}') f(\mathbf{x}')\, d^3x' \end{aligned} \tag{9-25}$$

This expression may be written in the form

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d^3x' \tag{9-26}$$

where the so-called *Green's function* is given by

$$G(\mathbf{x}, \mathbf{x}') = \sum_n \frac{u_n(\mathbf{x}) u_n^*(\mathbf{x}')}{\lambda_n - \lambda} \tag{9-27}$$

Note that a Green's function is determined by a differential operator (nearly always Hermitian), a particular region, and suitable boundary conditions. Sometimes one writes $G(\mathbf{x}, \mathbf{x}'; \lambda)$ to emphasize the dependence of G on λ as well as on $\mathbf{x}$ and $\mathbf{x}'$.

Let us find a differential equation obeyed by $G(\mathbf{x}, \mathbf{x}')$. Suppose $f(\mathbf{x})$ in the above derivation is taken to be $\delta(\mathbf{x} - \mathbf{x}_0)$. Then we obtain for the solution

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{x}')\, \delta(\mathbf{x}' - \mathbf{x}_0)\, d^3x = G(\mathbf{x}, \mathbf{x}_0)$$

Therefore, $G(\mathbf{x}, \mathbf{x}')$ is the solution of

$$LG(\mathbf{x}, \mathbf{x}') - \lambda G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \tag{9-28}$$

subject to the appropriate boundary conditions. This fact could also be recognized by applying the operator $L - \lambda$ to the infinite series representation (9-27) for $G(\mathbf{x}, \mathbf{x}')$ and using the completeness relation (9-14) (with $\rho = 1$).

The Green's function thus has a simple physical significance. It is the solution of the problem for a unit point "source" $f(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}')$.

We notice in (9-27) a feature which first appeared at the very beginning of this chapter. If λ equals an eigenvalue (so that a nontrivial solution of the *homogeneous* equation exists) there are serious difficulties involved in solving the inhomogeneous equation. When λ equals an eigenvalue λ_n, the Green's function (9-27) is infinite, and there is no solution $u(\mathbf{x})$ unless the right-hand side of the inhomogeneous equation obeys the condition

$$\int u_n^*(\mathbf{x}) f(\mathbf{x})\, d^3x = 0 \tag{9-29}$$

Let us consider a specific example, a string of length l vibrating with (angular) frequency ω. The equation and boundary conditions are

$$\frac{d^2u}{dx^2} + k^2u = 0 \qquad u(0) = u(l) = 0 \qquad k = \frac{\omega}{c} \tag{9-30}$$

What is the Green's function?

First method: Let $k^2 = -\lambda$

$$\frac{d^2u}{dx^2} = \lambda u$$

Eigenvalues are

$$\lambda_n = -\left(\frac{n\pi}{l}\right)^2 \qquad \text{with } n = 1, 2, 3, \ldots$$

Normalized eigenfunctions are

$$u_n = \sqrt{\frac{2}{l}} \sin \frac{n\pi x}{l}$$

Therefore, the general formula

$$G(x, x') = \sum_n \frac{u_n(x)u_n^*(x')}{\lambda_n - \lambda}$$

becomes

$$G(x, x') = \frac{2}{l}\sum_n \frac{\sin(n\pi x/l)\sin(n\pi x'/l)}{k^2 - (n\pi/l)^2} \tag{9-31}$$

The above result illustrates an important general symmetry relation for Green's functions, which follows from Eq. (9-27):

$$G(\mathbf{x}', \mathbf{x}) = [G(\mathbf{x}, \mathbf{x}')]^* \tag{9-32}$$

The physical significance of this result, for real Green's functions, is the *reciprocity* relation, that the response at $\mathbf{x}$ to a unit point disturbance at $\mathbf{x}'$ is the same as the response at $\mathbf{x}'$ to a unit point disturbance at $\mathbf{x}$.

Second method: Try solving the differential equation

$$\frac{d^2u}{dx^2} + k^2u = \delta(x - x') \tag{9-33}$$

which $G(x, x')$ obeys. In interpreting the physical significance of the result, it is useful to keep in mind that (9-33) is the time-independent equation for a vibrating string subject to a force $\propto -\delta(x - x')e^{-i\omega t}$. For x equal to anything but x',

$$\frac{d^2u}{dx^2} + k^2u = 0$$

Therefore,

$$G(x, x') = \begin{cases} a \sin kx & (x < x') \\ b \sin k(x - l) & (x > x') \end{cases} \tag{9-34}$$

How do we determine the constants a and b?

Integrate the differential equation (9-33), which $G(x, x')$ obeys, from $x' - \varepsilon$ to $x' + \varepsilon$, where ε is infinitesimal. The result is

$$\left.\frac{dG}{dx}\right|_{x'-\varepsilon}^{x'+\varepsilon} = 1 \tag{9-35}$$

Integrating again gives

$$G\Big|_{x'-\varepsilon}^{x'+\varepsilon} = 0 \tag{9-36}$$

That is, $G(x, x')$, as a function of x, is continuous at $x = x'$ but its first derivative jumps by $+1$ at that point. Using the expressions (9-34) for G, these conditions give

$$a \sin kx' = b \sin k(x' - l)$$

$$ka \cos kx' + 1 = kb \cos k(x' - l)$$

The solution of these simultaneous equations is

$$a = \frac{\sin k(x' - l)}{k \sin kl} \qquad b = \frac{\sin kx'}{k \sin kl}$$

and the Green's function is

$$G(x, x') = \frac{1}{k \sin kl} \begin{cases} \sin kx \sin k(x' - l) & 0 < x < x' \\ \sin kx' \sin k(x - l) & x' < x < l \end{cases}$$

or, in a more concise notation,

$$G(x, x') = \frac{-1}{k \sin kl} \sin kx_< \sin k(l - x)_< \tag{9-37}$$

where $\qquad x_< = \min(x, x') \qquad (l - x)_< = \min[(l - x), (l - x')]$

The same result is obtained if we solve

$$\frac{d^2u}{dx^2} + k^2 u = f(x)$$

by variation of parameters (see Problem 1-26). Note the same symmetry between x and x' that we obtained in (9-31).

Now let's work out a two-dimensional Green's function. We could just write down the formal sum involving normalized eigenfunctions, analogous to (9-31), but we shall use the second approach and solve the differential equation.

The problem is that of a circular drum, for which the equation and boundary conditions are

$$\nabla^2 u + k^2 u = 0, \qquad u = 0 \quad \text{when} \quad r = R \tag{9-38}$$

It is clear from physical considerations that $G(\mathbf{x}, \mathbf{x}')$ can depend only on r, r', and θ, the angle between $\mathbf{x}$ and $\mathbf{x}'$. Choose $\mathbf{x}'$ on the axis of polar coordinates, as shown in Figure 9-1.

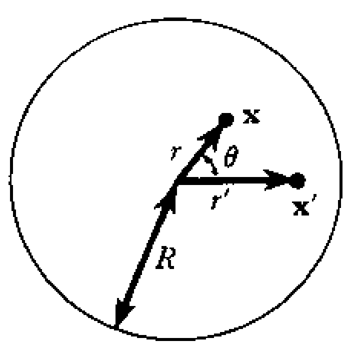

Figure 9-1 Coordinates for the circular-drum Green's function

Now G is the solution of

$$\nabla^2 G + k^2 G = \delta(\mathbf{x} - \mathbf{x}') \tag{9-39}$$

where $\delta(\mathbf{x} - \mathbf{x}')$ is the two-dimensional delta function; $\int \delta(\mathbf{x} - \mathbf{x}')\, d^2x = 1$ for any area of integration which includes $\mathbf{x}'$.

For $\mathbf{x} \neq \mathbf{x}'$,

$$\nabla^2 G + k^2 G = 0 \tag{9-40}$$

The solution of this equation satisfying the boundary conditions may be written in the form

$$G = \begin{cases} \sum_m A_m J_m(kr) \cos m\theta & (r < r') \\ \sum_m B_m [J_m(kr) Y_m(kR) - Y_m(kr) J_m(kR)] \cos m\theta & (r > r') \end{cases} \tag{9-41}$$

Note that we choose the solution for $r > r'$ so that it vanishes automatically at $r = R$; we have also used the physically obvious fact that G is an *even* function of θ.

We determine the constants A_m and B_m by fitting our solutions (9-41) together along the circle $r = r'$. As in our one-dimensional example, G is continuous but its derivative (that is, its *gradient*) has a discontinuity at the point $\mathbf{x} = \mathbf{x}'$. To find the singularity, integrate the differential equation (9-39) for G over an infinitesimal area which includes the point $\mathbf{x} = \mathbf{x}'$. We obtain

$$\int \nabla^2 G\, d^2x = \int (\nabla G)_n\, dl = 1 \tag{9-42}$$

where we have used the two-dimensional analog of Gauss' theorem:

$$\int_V \nabla \cdot \mathbf{u}\, d^3x = \int_\Sigma \mathbf{u} \cdot d\mathbf{S}$$

Σ being the surface which encloses the volume V.

For an area enclosing the point $\mathbf{x}'$, we shall choose one shaped as shown in Figure 9–2. Neglecting the ends, (9-42) gives us

$$\int_{r'+\epsilon} \frac{\partial G}{\partial r}\, dl - \int_{r'-\epsilon} \frac{\partial G}{\partial r}\, dl = 1$$

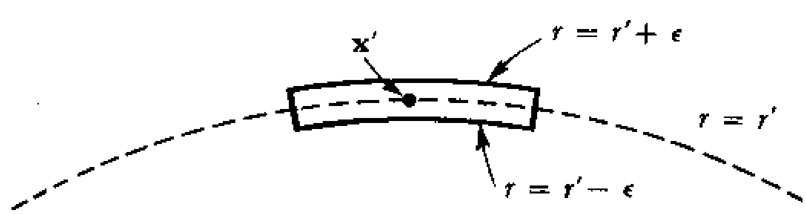

Figure 9–2 The area for application of Gauss' theorem

where l measures arc length. Since $dl = r'\,d\theta$, this gives

$$\int d\theta\left(\frac{\partial G}{\partial r}\bigg|_{r'+\varepsilon} - \frac{\partial G}{\partial r}\bigg|_{r'-\varepsilon}\right) = \frac{1}{r'}$$

(provided the range of integration includes the point $\mathbf{x}'$; otherwise we get zero). Therefore,

$$\left(\frac{\partial G}{\partial r}\bigg|_{r'+\varepsilon} - \frac{\partial G}{\partial r}\bigg|_{r'-\varepsilon}\right) = \frac{1}{r'}\,\delta(\theta)$$

Let

$$\frac{\partial G}{\partial r}\bigg|_{r'+\varepsilon} - \frac{\partial G}{\partial r}\bigg|_{r'-\varepsilon} = \sum_m C_m \cos m\theta$$

We multiply both sides by $\cos m'\theta$, in the usual way, and integrate over θ from $-\pi$ to π. The result is

$$\frac{1}{r'} = \pi C_{m'}\,\varepsilon_{m'} \qquad \text{where} \qquad \varepsilon_{m'} = \begin{cases} 2 & \text{if} \quad m' = 0 \\ 1 & \text{if} \quad m' > 0 \end{cases} \tag{9-43}$$

Thus

$$\frac{\partial G}{\partial r}\bigg|_{r'+\varepsilon} - \frac{\partial G}{\partial r}\bigg|_{r'-\varepsilon} = \frac{1}{\pi r'}\sum_m \frac{1}{\varepsilon_m}\cos m\theta \tag{9-44}$$

This is the condition on the discontinuity in the gradient of G that we were after.

Now we can write down two simultaneous equations for the A_m and B_m of (9-41). From the continuity of G at $r = r'$, we obtain

$$A_m J_m(kr') = B_m[J_m(kr')Y_m(kR) - Y_m(kr')J_m(kR)]$$

From the condition (9-44) on the discontinuity in $\partial G/\partial r$, we obtain

$$B_m[J_m'(kr')Y_m(kR) - Y_m'(kr')J_m(kR)] - A_m J_m'(kr') = \frac{1}{\pi\varepsilon_m kr'}$$

The solution is

$$A_m = \frac{J_m(kR)Y_m(kr') - J_m(kr')Y_m(kR)}{2\varepsilon_m J_m(kR)} \tag{9-45}$$

$$B_m = \frac{-J_m(kr')}{2\varepsilon_m J_m(kR)} \tag{9-46}$$

where we have used the relation (found in Problem 7-12)

$$J_m(x)Y_m'(x) - J_m'(x)Y_m(x) = \frac{2}{\pi x} \tag{9-47}$$

Note that $G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}', \mathbf{x})$, as before.

Let us now return to the original Green's function equation (9-39) and solve it by another method.

$$\nabla^2 G + k^2 G = \delta(\mathbf{x} - \mathbf{x}') \tag{9-48}$$

where $\delta(\mathbf{x} - \mathbf{x}')$ is the two-dimensional delta-function located at the "source" point $\mathbf{x}'$.

There are two difficulties in finding the solution: one is to insure the proper singularity at the source point $\mathbf{x}'$ and the other is to satisfy the boundary conditions. It is sometimes convenient to separate the two difficulties by finding a solution of the form

$$G = u(\mathbf{x}, \mathbf{x}') + v(\mathbf{x}, \mathbf{x}') \tag{9-49}$$

where u has the singularity at $\mathbf{x}'$ [that is, obeys Eq. (9-48)] without necessarily satisfying the boundary conditions, while v is "smooth" at $\mathbf{x}'$ [that is, obeys the homogeneous equation (9-40)] but is adjusted to make $G(\mathbf{x}, \mathbf{x}')$ satisfy the boundary conditions. The function $u(\mathbf{x}, \mathbf{x}')$ is sometimes called a *fundamental solution* of (9-48).

The singularity required at $\mathbf{x}'$ may be found by integrating Eq. (9-48) over a small area element centered at $\mathbf{x}'$ and finding u in the form $u(\rho)$ where $\rho = |\mathbf{x} - \mathbf{x}'|$. From Gauss' theorem [compare (9-42)]

$$\int_0^\rho \nabla^2 G \cdot 2\pi\rho \, d\rho = 2\pi\rho \frac{\partial G}{\partial \rho}$$

so that

$$2\pi\rho \frac{\partial G}{\partial \rho} + k^2 \int_0^\rho G \cdot 2\pi\rho \, d\rho = 1$$

and

$$G(\rho) \to \frac{1}{2\pi} \ln \rho + \text{constant} \qquad \text{as } \rho \to 0 \tag{9-50}$$

$u(\mathbf{x}, \mathbf{x}')$ should then have this behavior near $\rho = 0$. The singular solution of Eq. (9-40), $Y_0(k\rho)$, has the behavior for small ρ:

$$Y_0(k\rho) \to \frac{2}{\pi} \ln \rho + \text{constant}$$

Thus

$$G = \tfrac{1}{4} Y_0(k\rho) + v(\mathbf{x}, \mathbf{x}') \tag{9-51}$$

We now write

$$v = \sum_n A_n J_n(kr) \cos n\theta$$

with A_n chosen so that G satisfies the boundary conditions. That is,

$$G(r = R) = 0 = \tfrac{1}{4}Y_0(k\rho_R) + \sum_n A_n J_n(kR) \cos n\theta$$

$$A_n = -\frac{1}{4\pi J_n(kR)\varepsilon_n} \int_0^{2\pi} Y_0(k\rho_R) \cos n\theta \, d\theta$$

where

$$\rho_R^2 = R^2 + r'^2 - 2Rr' \cos \theta$$

(see Figure 9–3) and ε_n is defined in (9-43). The result is

$$G(\mathbf{x}, \mathbf{x}') = \tfrac{1}{4}Y_0(k\rho) - \sum_{n=0}^{\infty} \frac{J_n(kr) \cos n\theta}{2\pi J_n(kR)\varepsilon_n} \int_0^{\pi} Y_0(k\rho_R) \cos n\theta \, d\theta \qquad (9\text{-}52)$$

This form of the Green's function is convenient for some purposes; it is easy to visualize for low frequencies ω and source positions $\mathbf{x}'$ not too near the edge.

In summary, we recall the three forms found above for a Green's function:

1. The formal sum over eigenfunctions (9-27)
2. A solution of the homogeneous equation and boundary conditions on either side of a "surface" containing the source point; the two solutions being matched on this surface in such a way as to produce the source-point singularity
3. The form $G(\mathbf{x}, \mathbf{x}') = u(\mathbf{x}, \mathbf{x}') + v(\mathbf{x}, \mathbf{x}')$, where u is the fundamental solution, and v takes care of the boundary conditions.

If a problem is inhomogeneous because of the boundary conditions rather than the differential equation, the solution may still be written in terms of a Green's function. In fact, the examples in Chapter 8 of heat problems with initial temperature distributions were of this type, since initial conditions are boundary conditions in time (see also Problem 9-5).

Alternatively, a homogeneous equation with inhomogeneous boundary

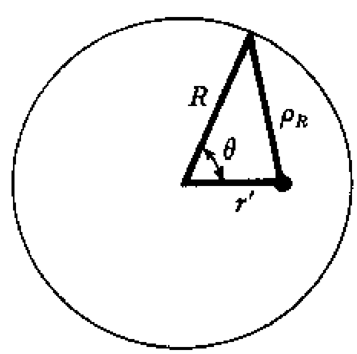

Figure 9–3 The coordinate ρ_R for the drum head

conditions may be transformed into an inhomogeneous equation with homogeneous boundary conditions. (The reverse is also true.) The transformation to accomplish this is not unique, and one should be careful to choose one which leads to a simple differential equation.

EXAMPLE

Consider the heat problem on p. 235. An infinite slab of thickness D and initial temperature zero has the surface $x = D$ insulated, and heat is supplied at the surface $x = 0$ at a constant rate, Q calories per sec per cm^2. The mathematical problem is

$$\frac{\partial^2 u(x, t)}{\partial x^2} - \frac{1}{\kappa}\frac{\partial u(x, t)}{\partial t} = 0 \tag{9-53}$$

$$u(x, 0) = 0 \qquad \left.\frac{\partial u}{\partial x}\right|_{x=D} = 0 \qquad \left.\frac{\partial u}{\partial x}\right|_{x=0} = -\frac{Q}{K}$$

where $\kappa = K/C\rho$, $K =$ thermal conductivity, $C =$ specific heat, and $\rho =$ density.

The problem may be transformed to one with homogeneous boundary conditions (in x) by a change of variable:

$$v(x, t) = u(x, t) - w(x)$$

where $w(x)$ satisfies the conditions

$$\left.\frac{dw}{dx}\right|_{x=D} = 0 \qquad \left.\frac{dw}{dx}\right|_{x=0} = -\frac{Q}{K}$$

and also $w(x)$ should be chosen so that the operation d^2w/dx^2 in the differential equation is easy to perform and gives a simple result. The simplest choice is probably the parabola

$$w(x) = \frac{Q}{2KD}(x - D)^2 \tag{9-54}$$

Then the differential equation (9-53) becomes

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{\kappa}\frac{\partial v}{\partial t} = -\frac{Q}{KD} \tag{9-55}$$

A particular solution is

$$v_p = \frac{Q\kappa}{KD}t \qquad u_p(x, t) = \frac{Q}{C\rho D}t + \frac{Q}{2KD}(x - D)^2 \tag{9-56}$$

which is the same solution we found before in (8-63).

As a final application of Green's functions, we observe that the use of a Green's function enables us to convert a partial differential equation into an integral equation. For example, consider the equation

$$\nabla^2 u(\mathbf{x}) = f(\mathbf{x})u(\mathbf{x}) \tag{9-57}$$

in some region Ω, with suitable boundary conditions. Let $G(\mathbf{x}, \mathbf{x}')$ be the Green's function of Laplace's equation[2] for the particular region and boundary conditions. Then

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \tag{9-58}$$

and the solution of

$$\nabla^2 u(\mathbf{x}) = g(\mathbf{x})$$

is

$$u(\mathbf{x}) = \int d^3x' \, G(\mathbf{x}, \mathbf{x}')g(\mathbf{x}') \tag{9-59}$$

Thus our original differential equation (9-57) is equivalent to the integral equation

$$u(\mathbf{x}) = \int d^3x' \, G(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')u(\mathbf{x}') \tag{9-60}$$

Note that the integral equation has the boundary conditions "built-in." A specific example of this application will be given in Chapter 11 [see Eq. (11-18)].

9-5 GREEN'S FUNCTIONS IN ELECTRODYNAMICS

We shall discuss briefly two Green's functions which are important in electrodynamics.

First, consider Laplace's equation

$$\nabla^2 \phi = 0 \tag{9-61}$$

in an infinite region. The boundary condition is that $\phi \to 0$ as $r \to \infty$. To find the Green's function, we must solve

$$\nabla^2 \phi(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}') \tag{9-62}$$

Clearly, ϕ can depend only on the scalar quantity $r = |\mathbf{x} - \mathbf{x}'|$. Thus we take our origin of (spherical) coordinates to be at the point $\mathbf{x}'$.

[2] We might equally well call $G(\mathbf{x}, \mathbf{x}')$ the Green's function for Poisson's equation. In general we may label a Green's function by either its homogeneous or inhomogeneous equation.