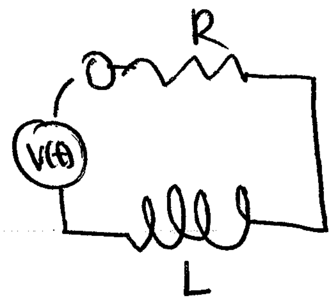
Application of Contour Integrals

Example 1 Consider a R-L circuit



Suppose we give an impulse

$$V(t) = A\delta(t) = \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,d\omega$$

$$\hat{V}(\omega) = \frac{A}{2\pi}$$

Well we know that

$$V(t) = L\frac{dI(t)}{dt} + I(t)R$$

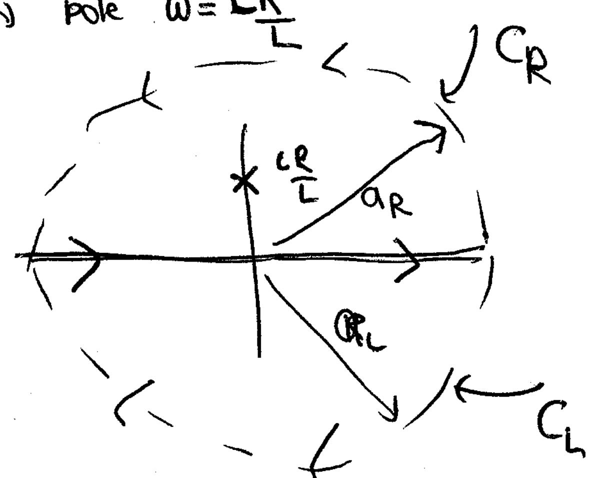
Fourier transforming

$$\hat{V}(\omega) = Li\omega\,\hat{I}(\omega) + R\,\hat{I}(\omega)$$

$$\Longrightarrow \quad \hat{I}(\omega) = \frac{A}{2\pi}\frac{1}{(Li\omega + R)}$$

$$I(t) = \frac{A}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}\,d\omega}{i\omega L + R}$$

pole $\omega = \frac{iR}{L}$



So now do by contour integration. Notice if $t > 0$ then in lower half plan $i\omega t$ has large negative part as $\mathrm{Im}\,\omega \to -\infty$

Thus, $\int_{C_L} \frac{e^{izt}}{izL + R}$ goes to zero $\left|\frac{e^{izt}}{izL+R}\right|\Big|_{C_L} \to 0$ as $a_L \to \infty$

Thus, for $t<0$, we close contour on the bottom and conclude

$$I(t) = \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{i\omega L + R} + \underbrace{\int_{C_L} f(z)dz}_{0} = 2\pi i \text{ Res} = 0$$

So
$$I(t) = 0 \quad \text{for } t<0.$$
<u>This must be true due to causality.!</u>

This is a general property of susceptibilities. Causality requires, that the be analytic in a half-plane! We will come back to this next lecture we discuss <u>dispersion-relations</u>.

For $t>0$, we close in the upper-half plane. Then,

$$|e^{i\omega t}| \to 0 \quad \text{as} \quad |\text{Im}\,\omega| \to \infty$$

So we know

$$\int_{C_R} f(z) \to 0$$

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{i\omega L + R} = 2\pi i \operatorname*{Res}_{\frac{iR}{L}} = 2\pi i\left(\frac{A}{2\pi}\right)\frac{e^{-Rt/L}}{iL} = \frac{A}{L}e^{-Rt/L}$$

as we expect!

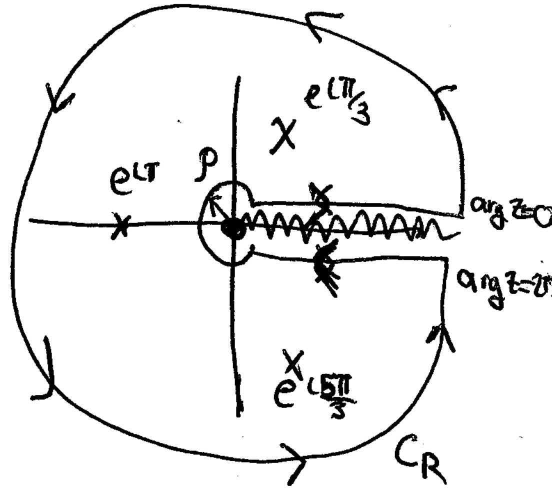
Example 2

$$I = \int_0^\infty \frac{dx}{1+x^3}$$

Notice this odd integral so cant extend to infinity. We will do this in some very long convoluted way to learn about integrating around branch points

Consider integral

$$\int_C \frac{\ln z}{1+z^3}$$



Notice that integral along C_R goes to zero since

$$\left|\frac{\ln z}{1+z^3}\right| \leq \left|\frac{\ln R}{R^3}\right| \quad \text{so} \quad \left|\int \frac{\ln z}{1+z^3}\right| \leq \pi \frac{\ln R}{R} \to 0 \quad \text{as } R \to \infty$$

Now consider integal along inner circle $z(\theta) = \rho e^{i\theta}$ $\quad 2\pi < \theta < 0$

$$\lim_{\rho \to 0} \; i\int_{2\pi}^{0} d\theta \, \rho \frac{[\ln \rho + i\theta]e^{i\theta}}{1+\rho^3 e^{3i\theta}} = 0$$

$\frac{dz}{d\theta}$ $\quad \frac{d}{d\theta}\rho e^{i\theta}$

Since integral vanishes.

Thus we only have Integrals above and below branch

Above branch $\longrightarrow \displaystyle\int_0^\infty \frac{dx \ln x}{1+x^2}$ $\quad$ below branch $\longrightarrow \displaystyle\int_\infty^0 dx \frac{\ln x + i2\pi}{1+x^3}$

$$= -\int_0^\infty \frac{\ln x}{1+x^3} dx - 2\pi i \int_0^\infty \frac{\ln x \, dx}{1+x^3}$$

So

$$\int_C \frac{\ln z}{1+z^3} = \overbrace{\cancel{\int_0^\infty \frac{dx \ln x}{1+x^2}}}^{\text{upper}} - \overbrace{\cancel{\int_0^\infty \frac{\ln x}{1+x^3}}}^{\text{lower}} - 2\pi i \int_0^\infty \frac{\ln x\, dx}{1+x^3}$$

cancel

$$2\pi i \sum \operatorname*{Res}_{z=z_0} = -2\pi i I$$

$z_0 = e^{i\pi/3}, e^{i\pi}, e^{i\frac{5\pi}{3}}$

$$I = \sum_{z_0 = e^{i\pi/3}, e^{i\pi}, e^{i5\pi/3}} \operatorname{Res}\left[\frac{\ln z}{1+z^3}\right]$$

$$\sum_{z_0 = e^{i\frac{\pi}{3}}, e^{i\pi}, e^{i\frac{5\pi}{3}}} \operatorname{Res} \frac{\ln z}{(z-e^{i\pi/3})(z-e^{i\pi})(z-e^{i5\pi/3})}$$

$$= \frac{\ln e^{i\pi/3}}{(e^{i\pi/3}-e^{i\pi})(e^{i\pi/3}-e^{i5\pi/3})} + \frac{\ln e^{i\pi}}{(e^{i\pi}-e^{i\pi/3})(e^{i\pi}-e^{i5\pi/3})}$$

$$+ \frac{\ln e^{i5\pi/3}}{(e^{i5\pi/3}-e^{i\pi/3})(e^{i5\pi/3}-e^{i\pi})}$$

$$= \left(\frac{2\pi\sqrt{3}}{9}\right)$$

<u>Example 5</u> Scattering In Quantum Mechanics and "Physical Meanings of ϵ- Prescription

(Due after next example)

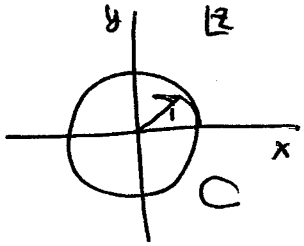
Example 3

Consider integrals of the form

$$I = \int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$$

$$z = e^{i\theta} \qquad dz = i e^{i\theta}\, d\theta$$

$$d\theta = -\frac{i\,dz}{z} \qquad \sin\theta = \frac{z - z^{-1}}{2i} \qquad \cos\theta = \frac{z + z^{-1}}{2}$$



$$I = -i \oint_C f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{dz}{z}$$

Consider

$$I = \int_0^{2\pi} \frac{d\theta}{1 + \varepsilon\cos\theta} \qquad |\varepsilon| < 1$$

$$= -i \oint \frac{dz}{z\left[1 + \frac{\varepsilon}{2}(z + z^{-1})\right]}$$

$$= -i\frac{2}{\varepsilon} \int \frac{dz}{z^2 + 2/\varepsilon\, z + 1} = -i\frac{2}{\varepsilon}\, 2\pi i \left.\frac{1}{z + \frac{1}{\varepsilon} + \frac{1}{\varepsilon}\sqrt{1-\varepsilon^2}}\right|_{z = z_+}$$

roots $z_- = -\frac{1}{\varepsilon} - \frac{1}{\varepsilon}\sqrt{1 - \varepsilon_2}$ ← outside circle

$$= \frac{2\pi}{\sqrt{1 - \varepsilon^2}}$$

$z_+ = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon}\sqrt{1 - \varepsilon^2}$ ← in circle

Example 4

Consider integral

$$\int_0^\infty dx \, \frac{\sin x}{x} \qquad \text{notince } \lim_{x \to 0} \frac{\sin x}{x} = 1$$

So well defined.

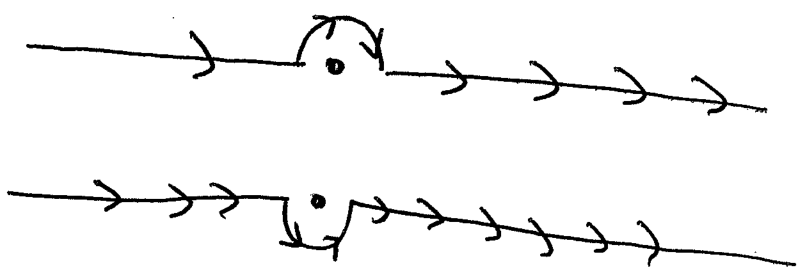
This is ~~imagi~~ imaginary part

$$I_z = P \int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz$$

Where P means "principal value"

Basic idea consider pole directly on contour integration

Can by pass pole



$$\int_{\pi}^{0} \frac{dx}{z - x_0} \leftarrow \text{Integration over semi-circle clockwise}$$

$$z = x_0 + \rho e^{i\theta}$$

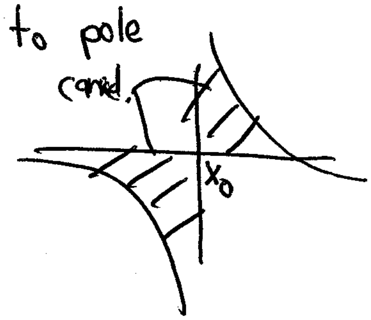
$$\int_{\pi}^{0} \left(\frac{d\theta}{\rho e^{i\theta}}\right)(\rho e^{i\theta}) i$$

$$= -i\pi$$

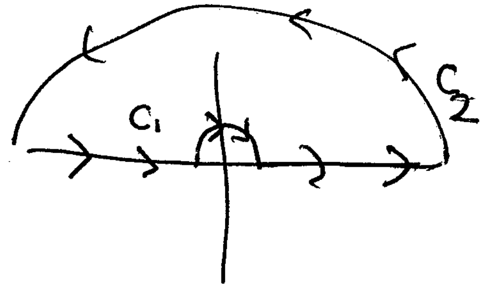
For counter clockwise $= -i\pi$.

Thus consider some function $f(z)$ with pole at x_0

$$f(z) \approx \frac{a_{-1}}{z - x_0} \leftarrow \text{close enough to pole}$$



$$P \int_{-\infty}^{\infty} f(x) dx = \lim_{a \to \infty} \int_{-a}^{a} f(x) dx$$



So now

$$\oint \frac{e^{iz}dz}{z} = P\int_{-\infty}^{\infty} \frac{e^{ix}}{x} + \int_{C_2} \frac{e^{iz}}{z} + \int_{C_1} \frac{e^{iz}}{z}$$

0 (no poles)

So now $$\int_{C_1} \frac{e^{iz}}{z} = \pi i$$

Furthermore, $$\int_{C_2} \frac{e^{iz}}{z} = 0$$ by <u>Jordan's Lemma</u>

a) $f(z)$ is analytic in upper half plane except finite number of poles

b) $\lim_{|z|\to\infty} f(z) = 0 \qquad 0 \le \arg z \le \pi$

Then $\int_{C_2} f(z)dz = 0$. (See Arfken p.424 for proof)

$$\Longrightarrow P\int_{-\infty}^{\infty} \frac{e^{ix}}{x} = \pi i \Longrightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x} = \pi$$

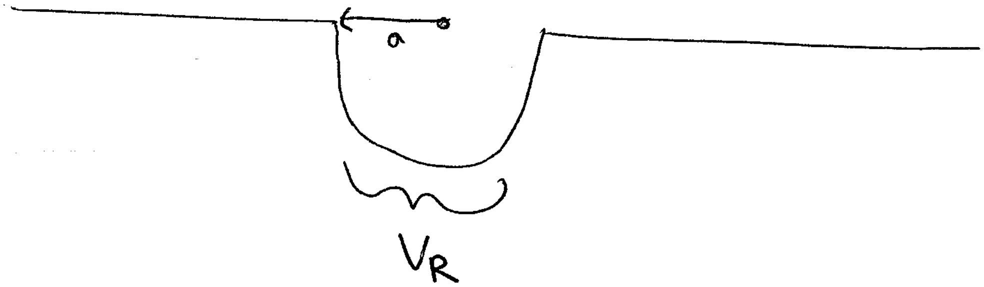
Now for some physics

Physics Example 1

Scattering theory in Q.M.

Consider a localized potential

e^{ikr} $\longrightarrow$



$r \to -\infty$ corresponds to far past

$r \to \infty$ corresponds to far future

Consider ~~[illegible]~~ plane wave e^{ikr} (traveling away from origin)

Well then one will in general get scattered outgoing wave far away from the scattering region must have form

$$e^{ik \cdot r} + \underbrace{f_k(\theta, \varphi)}_{\text{This is scattering amplitude}} \frac{e^{ikr}}{r}$$

(ie plane wave that decay's at infinity)

So look at Schrondiger's equation

$$\left(\frac{\hbar^2 \nabla^2}{2m} + E_k\right)\Psi_k(\vec{r}) = V(\vec{r})\Psi_k(r) \qquad E_k = \frac{\hbar^2 k^2}{2m}$$

Then

$$\Psi_k(\vec{r}) = e^{ikr} + \int dr'\, G(r - r')\left[V(r')\Psi_k(r')\right]$$

where $G(r - r')$ is Green's Function satisfying

$$\left(\frac{\hbar^2 \nabla^2}{2m} + E_k\right) G_{R,k} = \delta(\vec{r})$$

[like inverse matrix]

(notice add e^{ikr} and still solution)

Basic idea of Green's Function

$$L = \frac{\hbar^2 \nabla^2}{2m} + E_k$$

$$L\, \psi_k(\vec{r}) = g(x)$$

$$\psi_k(\vec{r}) = L^{-1} g(x)$$

Spend all of October doing this in detail!

One finds

$$G(r,k) = \overbrace{\int \frac{d^3k'}{(2\pi)^3} \frac{e^{ik'\cdot r}}{E_k - \frac{\hbar^2 k'^2}{2m}}}^{\text{Inverse Fourier Transform}} = \frac{-m}{2\pi^2 i\hbar^2} \int_{-\infty}^{\infty} \frac{k'\,dk'\,e^{ik'r}}{k'^2 - k^2}$$

here $k\cdot r = \cos\theta$

$d^3k = d\varphi\, d(\cos\theta)\, k\, dk$

So now we have the contour integral

$$\int_{-\infty}^{\infty} \frac{k'\,dk'\,e^{ik'r}}{k'^2 - k^2}$$

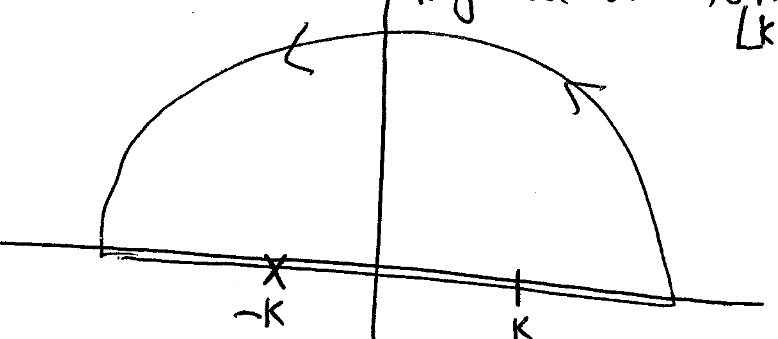
This is <u>not</u> well defined.

Poles are on <u>real axis</u>

What should we do?

Use physics. ⟶ Physics is actually hidden in these thing we do to make mathematic well-defined!

ik'



We can complete contour in upper-half plane since $r > 0$ by Jordans-Lemma

this vanishes on infinite semi-circle

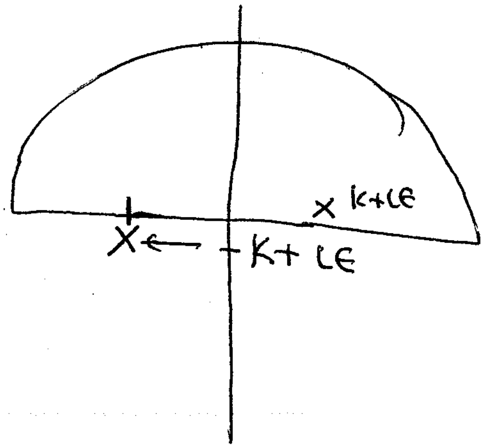
So now we know we want <u>outgoing plane wave</u> ie. e^{ikr} not ~~incoming~~ e^{-ikr} solution

$\leftarrow (e^{i(kr-\omega t)})$ $\quad e^{-i(kr+\omega t)}$ (in coming wave)

and definitely not both so we should

$e^{ikr} + e^{-ikr} \sim \cos kr$ ⟵ standing waves

This gives us perscription → So enclose pole at $+k$ but not at $-k$



So then call this

$$G_+(r,k) = \frac{-m}{2\pi^2 i\hbar^2}\int_{-\infty}^{\infty} dk' \, \frac{k' e^{ik'r}}{k'^2 - k^2 - i\epsilon}$$

$$= \frac{-m}{2\pi i\hbar^2}\int_C \frac{k' e^{ik'r}}{[k'-(k+i\epsilon)][k'-(-k-i\epsilon)]}$$

$$G_+(r,k) = \frac{m}{\hbar^2}\frac{e^{ikr}}{2}$$

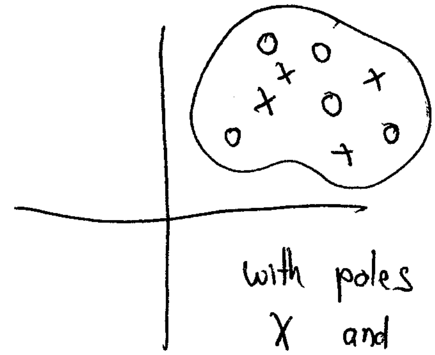
Physics Example 2

Winding Numbers, Argument Principle, Berry's Phase and Polarization in 1-D materials.

Argument Principle + Winding Number

Consider some function

$f(z)$ which has finite number of poles and singularities in region C



with poles X and zeros O

$$\oint_C \frac{f'(z)}{f(z)} = 2\pi i (N - P)$$

N = # of zeroes (counted with multiplicity)
P = # of poles (counted with order)

Proof

Let z_n be zero of f. $\quad f(z) = (z - z_N)^k g(z)$ so $g(z_n) \neq 0$.

$$f'(z) = k(z - z_N)^{k-1} g(z) + (z - z_N)^k g'(z)$$

$$\frac{f'(z)}{f(z)} = \frac{k}{(z - z_N)} + \frac{g'(z)}{g(z)}$$

$\dfrac{g'(z)}{g(z)}$ is analytic at z_n So

$$\operatorname*{Res}_{z = z_n} \frac{f'(z)}{f(z)} = k_N$$

Let z_p be a pole of f. Write $f(z) = (z - z_p)^{-m} h(z)$ where $h(z_p) \neq 0$. Then

$$f'(z) = -m(z - z_p)^{-m-1} h(z) + (z - z_p) h'(z)$$

$$\frac{f'(z)}{f(z)} = \frac{-m}{z - z_p} + \left.\frac{h'(z)}{h(z)}\right\}$$

no singularities at z_p since $h(z_p) \neq 0$

So $$\operatorname*{Res}_{z = z_p} \frac{f'(z)}{f(z)} = -m_p$$

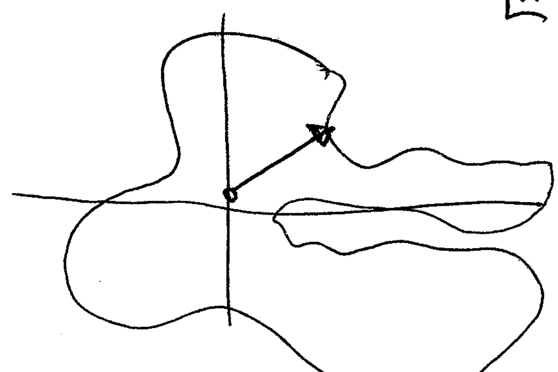
Thus, by residue Theorem

$$\int_C \frac{f'(z)}{f(z)} = 2\pi i (N - M)$$

However, notice

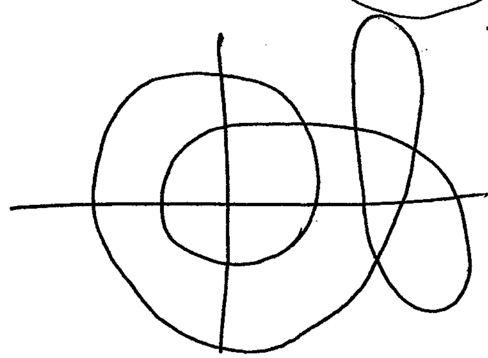
$$\oint_C dz \, \frac{f'(z)}{f(z)} = \ln f(z) = L\arg_c(f)$$

Since $2\pi L\arg_c(f)$ is just "winding number" $\times 2\pi L$ of $f(z)$ since ln measures how many times $w = f(z)$ wraps the origin in W plane



So we have that winding number

$$I(C,0) = (N-P)$$



The "winding number" is a topological quantum number → it is quantized in units of integres. ie. $N-P$

This is like topological charge. → It depends only on number of poles and zeros not on any other detals of $f(z)$. Create surfaces where you puncture "holes" with + charge and poles with "−" charge in complex plane.

This is really amazing!!

We just heard about topological insulators what can we do with this case.