**Application of Contour Integrals**

Example 1 Consider a R-L circuit

Suppose we give an impulse

\[ V(t) = A \delta(t) \]

\[ = \frac{A}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \, dw \]

\[ \hat{V}(\omega) = \frac{A}{2\pi} e^{i\omega t} \]

Well we know that

\[ V(t) = L \frac{dI(t)}{dt} + I(t)R \]

Fourier Transforming

\[ \hat{V}(\omega) = L \omega \hat{I}(\omega) + R \hat{I}(\omega) \]

\[ \Rightarrow \hat{I}(\omega) = \frac{A}{2\pi} \frac{e^{i\omega t}}{(L\omega + R)} \]

\[ I(t) = \frac{A}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t} \, dw}{(L\omega + R)} \]

So now do by contour integration. Notice if \( \omega \) is negative, then in lower half plane contour integral has large negative part as \( \text{Im} \omega \rightarrow -\infty \)

Thus, \[ \int_{C_L} \frac{e^{i\omega t}}{(L\omega + R)} \rightarrow 0 \] as \( R \rightarrow \infty \)
Thus, for \( t \leq 0 \), we close \( \text{contour} \) on the bottom and conclude

\[
I(t) = \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{i\omega L + R} + \int_{C_L} f(z) \, dz = 2\pi i \text{Res} = 0
\]

So

\[
I(t) = 0 \quad \text{for} \quad t < 0. \quad \text{This must be due to causality!}
\]

This is a general property of susceptibilities (causality requires, that the be analytic in a half-plane). We will come back to this next lecture we discuss dispersion relations.

For \( t > 0 \), we close in the upper-half plane. Then,

\[
e^{i\omega t} \to 0 \quad \text{as} \quad |\text{Im} \, \omega| \to \infty
\]

So we know

\[
\int_{C_R} f(z) \to 0
\]

\[
\int_{-\infty}^{\infty} e^{i\omega t} = 2\pi i \text{Res} = 2\pi i \left( \frac{A}{2\pi i} \right) \frac{e^{-R t L}}{LL} = \frac{A e^{-R t L}}{L}
\]

as we expect!
Example 2

\[ I = \int_0^\infty \frac{dx}{1 + x^3} \]

Notice this odd integral so can't extend to infinity. We will do this in some very long convoluted way to learn about integrating around branch points.

Consider integral

\[ \int_C \frac{\ln z}{1 + z^3} \]

Notice that integral along \( C_R \) goes to 0 as \( R \to \infty \) since

\[ \left| \frac{\ln z}{1 + z^3} \right| \leq \left| \frac{\ln R}{R^3} \right| \]

so

\[ \int_C \frac{\ln z}{1 + z^3} \leq \frac{\pi \ln R}{R} \to 0 \quad \text{as} \quad R \to \infty \]

Now consider integral along inner circle \( \theta \neq 0 \), \( pe^{i\theta} \), \( \pi \theta < 0 \)

\[ \lim_{\rho \to 0} \int_0^{2\pi} i\frac{e^{i\theta}}{1 + pe^{3i\theta}} d\theta = 0 \]

Since integral vanishes.

Thus we only have integrals above and below branch.

Above branch

\[ \int_0^\infty \frac{d\ln x}{1 + x^2} \]

below branch

\[ \int \frac{dx}{1 + x^2} \]

\[ = -\int \frac{d\ln x}{1 + x^3} dx - 2\pi i \int \frac{d(\frac{1}{x})}{1 + x^3} \]

\[ = -\int \frac{d\ln x}{1 + x^3} dx - 2\pi i \int \frac{dx}{1 + x^3} \]
So
\[ \int_{C} \frac{\ln z}{1 + z^3} = \int_{0}^{\infty} \frac{dx}{x} \frac{\ln x}{1 + x^3} - \int_{0}^{\infty} \frac{\ln x}{1 + x^3} = -2\pi i \int_{0}^{\infty} \frac{\ln x \, dx}{1 + x^3} \]
\[2\pi i \sum \text{Res} \quad \begin{array}{c} z = z_0 \end{array} \]
\[z_0 = e^{\pi i 3}, e^{\frac{\pi i}{3}}, e^{\frac{5\pi i}{3}} \]
\[I = \sum \text{Res} \left\{ \frac{\ln z}{1 + z^3} \right\} \]
\[= \sum \text{Res} \frac{\ln z}{(z - e^{\pi i 3})(z - e^{\frac{\pi i}{3}})(z - e^{\frac{5\pi i}{3}})} \]
\[= \frac{\ln e^{\pi i 3}}{(e^{\pi i 3} - e^{\frac{\pi i}{3}})(e^{\pi i 3} - e^{\frac{5\pi i}{3}})} + \frac{\ln e^{\frac{\pi i}{3}}}{(e^{\pi i 3} - e^{\frac{\pi i}{3}})(e^{\frac{5\pi i}{3}} - e^{\frac{\pi i}{3}})} + \frac{\ln e^{\frac{5\pi i}{3}}}{(e^{\pi i 3} - e^{\frac{5\pi i}{3}})(e^{\frac{5\pi i}{3}} - e^{\frac{\pi i}{3}})} \]
\[= \frac{2\pi i \sqrt{3}}{q} \]

Example: Scattering in Quantum Mechanics and Physical Meanings of E- prescription (Due after next example)
Example 3

Consider integrals of the form

\[ I = \int_0^{2\pi} f(\sin \theta, \cos \theta) \, d\theta \]

\[ z = e^\theta \quad dz = e^\theta \, d\theta \]

\[ d\theta = -\frac{dz}{z} \quad \sin \theta = \frac{z - z^*}{2i} \quad \cos \theta = \frac{z + z^*}{2} \]

\[ I = -C \int_C f \left( \frac{z - z^*}{2i}, \frac{z + z^*}{2} \right) \frac{dz}{z} \]

Consider

\[ I = \int_0^{2\pi} \frac{d\theta}{1 + e \cos \theta} \quad |e| < 1 \]

\[ = -C \int_C \frac{dz}{z \left[ 1 + \frac{z^2}{2} (z + z^*) \right]} \]

\[ = -\left( \frac{2}{e} \right) \int \frac{dz}{z^2 + \frac{2}{e} z + 1} = -\left( \frac{2}{e} \right) \left[ 2\pi i \frac{1}{z + \frac{1}{e} + \frac{1}{e\sqrt{1-e^2}}} \right] \]

\[ \text{root} \quad \frac{z}{z^-} = -\frac{1}{e} - \frac{1}{e} \sqrt{1 - e^2} \quad \text{outside circle} = \frac{2\pi}{\sqrt{1 - e^2}} \]

\[ z_+ = -\frac{1}{e} + \frac{1}{e} \sqrt{1 - e^2} \quad \text{in circle} \]
Thus consider some function \( f(z) \) with pole at \( z_0 \).

For counter clockwise close enough to pole

\[
\lim_{x \to 0} \int_{-\infty}^{\infty} \frac{x}{\pi(x^2 + 1)} \, dx
\]

Basic idea consider pole directly on contour integration

\[
\int_{-\infty}^{\infty} \frac{x}{x^2 + 1} \, dx
\]

Can by pass pole

where \( P \)

means "principal value"

Consider integral

This is imaginary part

Also well defined

\[
\int_{0}^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2}
\]

\[
\int_{0}^{\infty} \frac{\cos x}{x} \, dx = \frac{\pi}{2}
\]

\[
\int_{0}^{\infty} \frac{\ln \sin x}{x} \, dx = \frac{\pi}{2}
\]

\[
\int_{0}^{\infty} \frac{\ln x}{x} \, dx
\]

Example 4
So now
\[
\int_0^\infty \frac{e^{\alpha z} \, dz}{z} = 0 \quad (\text{no poles})
\]

So now \( \int_{c_1} \frac{e^{\alpha z}}{z} \, dz = \pi i \)

Furthermore, \( \int_{c_2} \frac{e^{\alpha z}}{z} \, dz = 0 \) by Jordan's lemma

a) \( f(z) \) is analytic in upper half plane except finite number of poles

b) \( \lim_{|z| \to \infty} f(z) = 0 \) \( 0 \leq \arg z \leq \pi \)

Then \( \int_{c_2} \frac{f(z)}{z} \, dz = 0 \). (See Ahlfors p.424)

\( \Rightarrow \quad \pi \int_{-\infty}^\infty \frac{e^{\alpha x}}{x} \, dx = \pi i \quad \Rightarrow \quad \int_{-\infty}^\infty \frac{\sin x}{x} \, dx = \pi \)
Now for some physics

Scattering theory in Q.M.

Consider a localized potential $V(r)$

Consider a plane wave $e^{ikr}$ (traveling away from origin)

Well then one will in general get scattered outgoing wave far away from the scattering region must have form

$$e^{ikr} + \frac{f(\theta, \phi)}{kr} e^{ikr}$$

(ie plane wave that decays at infinity)

This is scattering amplitude

So look at Schrödinger's equation

$$\left(\frac{-\hbar^2}{2m} \nabla^2 + E_k\right)\psi_k(\mathbf{r}) = V(\mathbf{r})\psi_k(\mathbf{r}) \quad E_k = \frac{\hbar^2 k^2}{2m}$$

Then

$$\psi_k(\mathbf{r}) = e^{ikr} + \int d\mathbf{r}' G(\mathbf{r} - \mathbf{r}') [V(\mathbf{r}')\psi_k(\mathbf{r}')]$$

where $G(\mathbf{r} - \mathbf{r}')$ is Green's Function satisfying

$$\left(\frac{-\hbar^2}{2m} \nabla^2 + E_k\right)G_{R,k} = \delta(\mathbf{r})$$

[Like inverse matrix]

(notice add $e^{ikr}$ and still solution)
Basic idea of Green's function

\[ L = \frac{\hbar^2 \nabla^2}{2m} + E_k \]

\[ \hat{L} \Psi_k(\vec{r}') = g(x) \]

\[ \Psi_k(\vec{r}) = \hat{L}^{-1} g(x) \]

Spend all of October doing this in detail!

One finds

\[ G(\vec{r}, k) = \int \frac{d^3k'}{(2\pi)^3} \frac{e^{ik'r}}{E_k - k^2} \]

\[ = -\frac{m}{2\pi^2 \hbar^2} \int_{-\infty}^{\infty} \frac{K' dk' e^{iK'r}}{K'^2 - K^2} \]

So now we have the contour integral

\[ \int_{-\infty}^{\infty} \frac{K' dk' e^{iK'r}}{K'^2 - K^2} \]

This is not well defined, poles are on real axis

What should we do?

Use physics. Physics is actually hidden in these things we do to make mathematic well-defined!

We can complete contour in upper-half plane since \( r > 0 \)
by Jordan's Lemma

\[ e^{iK'r} \text{ solution} \]
\[ e^{-iK'r} \text{ solution} \]
\[ \text{incoming wave} \]
\[ \text{standing waves} \]
This gives us perscription \( \rightarrow \) so enclose pole at \(+k\)

but not at \(-k\)

\[ G_+ (r, k) = \frac{\pi}{2\pi i \hbar^2} \int_{c} k' \frac{e^{ik'r}}{k'^2 - k^2 - \hbar \epsilon} \]

\[ C_+ (r, k) = \frac{\hbar}{\hbar^2} \frac{e^{i k r}}{2} \]

**Physics Example 2**

Winding Numbers, Argument Principle, Berry's Phase

and Polarization in 1-D materials.

Argument Principle + Winding Number

Consider some function

\[ f(z) \text{ which has finite number of poles and singularities in region } C \]

\[ \oint_C \frac{f'(z)}{f(z)} = 2\pi i (N - P) \]

\( N = \# \text{ of zeroes (counted with multiplicity)} \)

\( P = \# \text{ of poles (counted with order)} \)
Proof

Let $z_n$ be a zero of $f$. Then $f(z) = (z-z_n)^k g(z)$ so $g(z_n) \neq 0$.

\[
f'(z) = k(z-z_n)^{k-1} g(z) + (z-z_n)^k g'(z)
\]

\[
\frac{f'(z)}{f(z)} = \frac{k}{z-z_n} + \frac{g'(z)}{g(z)}
\]

$\frac{g'(z)}{g(z)}$ is analytic at $z_n$. So

\[
\text{Res}_{z=z_n} \frac{f'(z)}{f(z)} = k_n
\]

Let $z_p$ be a pole of $f$. Write $f(z) = (z-z_p)^{-m} h(z)$ where $h(z_p) \neq 0$. Then

\[
f'(z) = -m (z-z_p)^{-m-1} h(z) + (z-z_p)^{-m} h'(z)
\]

\[
\frac{f'(z)}{f(z)} = -\frac{m}{z-z_p} + \frac{h'(z)}{h(z)} \quad \text{no singularities at } z_p, \quad h(z_p) \neq 0
\]

So

\[
\text{Res}_{z=z_p} \frac{f'(z)}{f(z)} = -m_p
\]

Thus, by Residue Theorem

\[
\int_{\gamma} \frac{f'(z)}{f(z)} = 2\pi i (N-M)
\]
However, notice

\[ \oint_C \frac{f'(z)}{f(z)} = \ln f(z) = \text{Log}_c(f) \]

Since \( \text{Log}_c(f) \) is just "winding number" \( \times 2\pi i \) of \( f(z) \), since \( \ln \) measures how many times \( Wf(z) \) wraps the origin in \( W \) plane

So we have that winding number

\[ W(C,0) = (N - P) \]

The "winding number" is a topological quantum number \( \to \) it is quantized in units of integers \( \text{i.e.} \ N - P \)

This unlike topological charge. \( \to \) It depends only on number of poles and zeros not on any other details of \( f(z) \). Create surfaces where you puncture "holes" with + charge and poles with "-" charge in complex plane.

This is really amazing!! We just heard about topological insulators what can we do with this case.