VII Series (review)

In order to proceed, we have to generalize the Taylor series to something called the Laurent series.

In this section, we briefly review some basic properties of series as they apply to complex numbers.

An infinite series

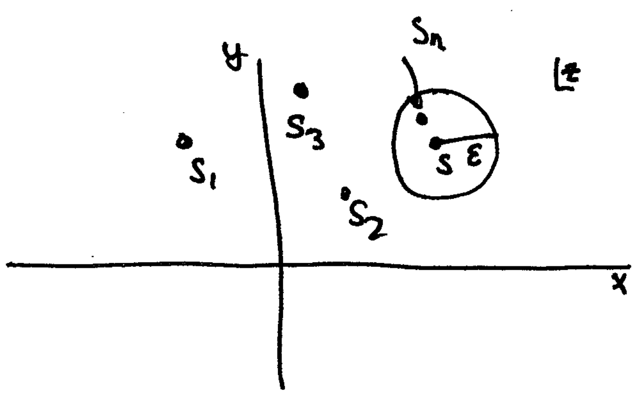
$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots$$

converges to the sum S if the sequence

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N$$

of partial sums converges to S.

$$\lim_{N\to\infty} S_N = S \quad \Big(\text{ie. } \forall \varepsilon>0, \exists n_0 \text{ s.t. } |S_N - S| < \varepsilon \text{ whenever } n > n_0$$



(i.e.) The partial sums get closer and closer to S as $n \to \infty$.

Theorem

Suppose $z_n = x_n + i y_n$ and $S = X + iY$, then

$$\sum_{n=1}^{\infty} z_n = S$$

if and only if $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$.

This theorem means that both the real and imaginary parts seperately converge as real sums.

A complex series is said to be absolutely convergent if the series

$$\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2}$$ of real

numbers converges.

Corollary The absolute convergence of a series of complex numbers implies the convergence of that series.

Useful to define remainder ρ_N after N terms

$$\rho_N = S - S_N.$$

Since $S = S_N + \rho_N$ and $|S_N - S| = |\rho_N|$

A series converges to a number S iff the sequence of remainders tends to zero.

Example

Verify $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ whenever $|z|<1$

$$S_N(z) = \sum_{n=0}^{N} z^n = 1+z+z^2 \cdots + z^N = \frac{1-z^{n+1}}{1-z}.$$

$$\rho_N(z) = -S_N(z)+S(z) = \frac{z^n}{1-z}$$

So $\rho_n(z) = \frac{|z|^N}{|1-z|}$

as $N\to\infty$ we know $|z|^N \to \begin{cases} 0 & \text{if } |z|<1 \\ \infty & \text{if } |z|>1. \end{cases}$

Thus, series converges for $|z|<1$.

VIII Taylor Series + Laurent Series

We now generalize the familiar notion of a Taylor series.

Theorem Suppose that a function f is analytic throughout a disk $|z-z_0|<R_0$, centered at z_0 and with radius R_0. Then $f(z)$ has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \qquad (|z-z_0|<R) \qquad (1)$$

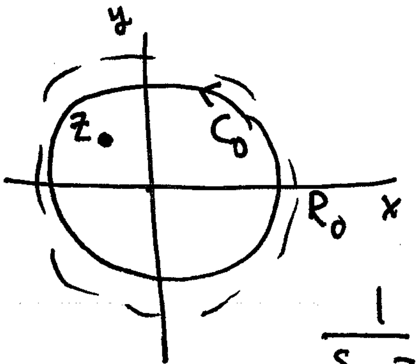
where $a_n = \frac{f^n(z_0)}{n!} = \frac{1}{2\pi i}\int \frac{f(z)}{(z-z_0)^{n+1}}$ $n = 0, 1, 2$

That is series (1) converges to $f(z)$ when z lies in the stated open disk.

Called Maclurin series

Proof We prove for the case $z_0 = 0$. The general proof is almost same

So now write



$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{s - z}$$

Now notice $|z/s| < 1$ from picture

$$\frac{1}{s-z} = \frac{1}{s} \frac{1}{(1 - z/s)}$$

$$= 1 + \left(\frac{z}{s}\right) + \left(\frac{z}{s}\right)^2 + \cdots$$

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2\pi i} \int_{C_0} \frac{f(s)}{s^{n+1}}$$

$$f(z) = \sum_{n=0}^{\infty} z^n \frac{f^n(0)}{n!}$$

Derivative formula

(physics "proof")

(

So we have proven Taylor series exists Using Contour integration.

The Taylor expansions are same as before. However, the don't always exists

Example

$$f(z) = \frac{1+2z^2}{z^3+z^5}$$

Here we see there is a singularity at zero. Let us "factor it out"

$$f(z) = \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right)$$

This has good Taylor series around zero ($z=0$)

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 \dots$$

$$f(z) = \frac{1}{z^3} + \frac{1}{z} - z + z^3 - z^5 + \cdots$$

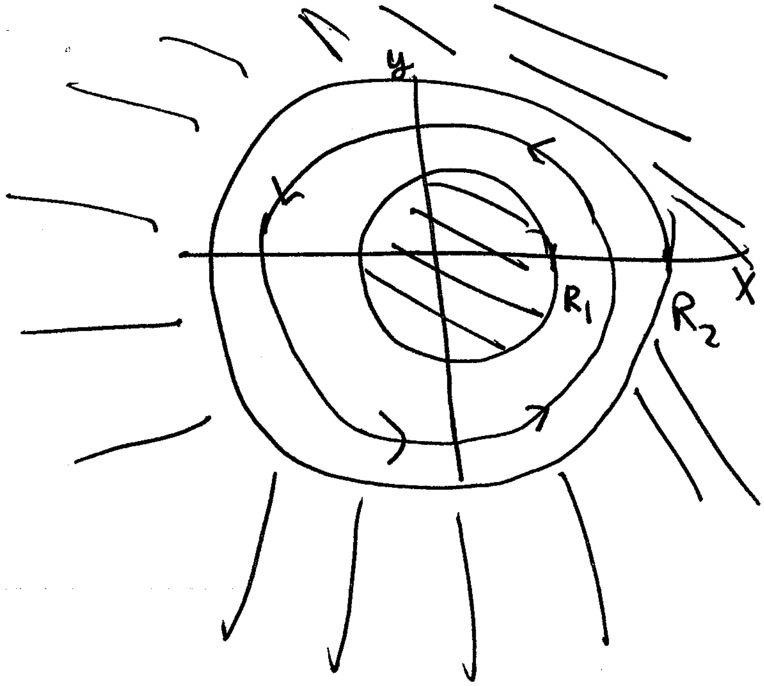
Notice that "negative powers" of z appear.

Such expansions where negative powers appear are called Laurent series!

Theorem Suppose a function f is analytic throughout an annular domain $R_1 < |z-z_0| < R_2$ centered at z_0, and let C denote any positively oriented simple contour around z_0 in that domain, Then at each point in the domain, $f(z)$ has a series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad (R_1 < |z-z_0| < R_2)$$

$\Rightarrow$



where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

$$b_n = \frac{1}{2\pi i} \int_C (z-z_0)^{n-1} f(z)\,dz$$

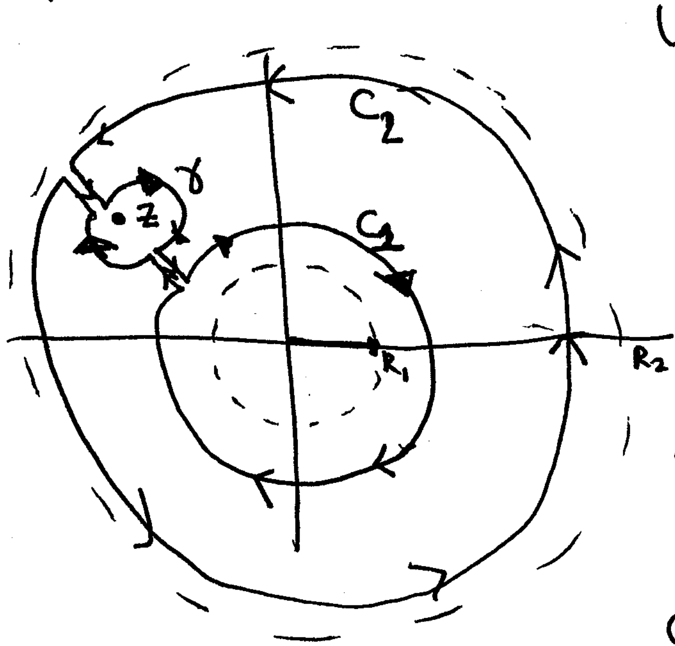
Alternatively, we can write

Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \quad (n = 0, \pm 1, \pm 2, \pm 3, \dots)$$

Proof



Using the contour shown on left we can conclude

$$\int_{C_2} \frac{f(\xi)\,ds}{z-s} - \int_{C_1} \frac{f(s)}{s-z} - \int_{\gamma} \frac{f(s)\,ds}{\xi - z}\,dz = 0$$

Where C_2 is outer circle with c.c.w. orientation.

C_1 is ~~outer~~ inner circle with c.c.w orientation (opposite of diagram)

and γ is circle around z_0 with ccw orientation (opposite of diagram)

We know that

$$\int_\gamma \frac{f(s)\,ds}{s-z} = 2\pi i f(z)$$

So we have

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\,ds + \frac{1}{2\pi i}\int_{C_1} \frac{f(s)\,ds}{(z-s)}$$

switched signs

Notice for C_2 $\left|\frac{z}{s}\right| < 1$

whereas for C_1 $\left|\frac{s}{z}\right| < 1$. So we can just do same trick as befor and expand

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{z}{s}\right)^n ds + \frac{1}{2\pi i}\int_{C_1} \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{s}{z}\right)^n ds$$

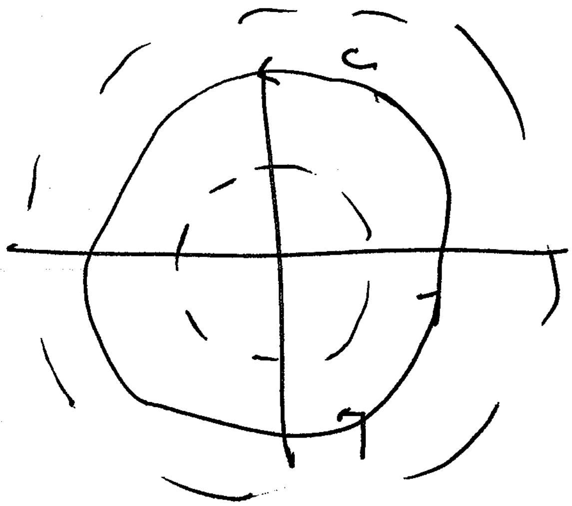
$$\cancel{= \frac{1}{2\pi i}\sum_{n=0}^{\infty} z^n a_n + \sum_{n=1}^{\infty}\frac{1}{2\pi i} z^{-n}\int_{C_1}\frac{f(s)}{s^{-n+1}} =}$$

$$\cancel{a_n = \int_{C_2}\frac{ds}{s^{n+1}}}$$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

$$a_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(s)\,ds}{s^{n+1}} \qquad b_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(s)\,ds}{s^{-n+1}}$$

Using the fact that integral around any curve in annulus is same, we can replace C_1 and C_2 by generic curve C in region to complete proof!



Example Replacing z by $\frac{1}{z}$ in the expansion

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z}{2!} + \cdots$$

We have

$$e^{1/z} = 1 + \frac{1}{1!\,z} + \frac{1}{2!\,z^2} + \cdots\cdots$$

We know that coeffierent on "negtative power" terms are

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} \qquad n = 1, 2, \ldots$$

Where $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

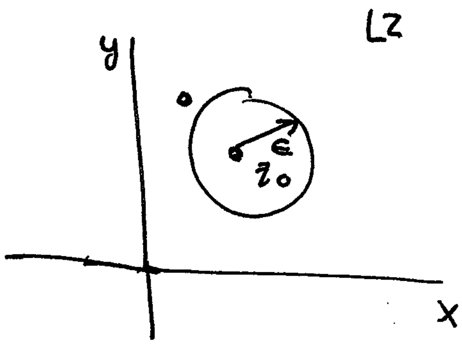
Thus, integral $\int_C f(z) = 2\pi i (b_1) = 2\pi i$

This is example of something called "residue" theorem we will revisit in a few pages.

IX Singularities : Poles + Branch Cuts + Residues

Residues+

A point z_0 is called a singular point of a function if f fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0. An <u>isolated singular point</u> is a singular point that meets the additional criterion that there is a deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0 throughout f is analytic.



In other words, there are no singularities arbitrarily close to z_0.

These kind of distinctions become important when one is dealing with fractal structures!.

Example 1

$$f(z) = \frac{1}{z^2+1}, \qquad f'(z) = \frac{-2z}{(z^2+1)}$$

This clearly has isolated singular points at $z = \pm i$ because derivative does not exist there but exists for every neighborhood of $z = \pm i$.

Example 2

Consider the multi-valued function $f(z) = \log z$.

Now we can write

$$\log z = r + i(\theta + 2\pi i n) \qquad n = 0, \pm 1, \pm 2$$

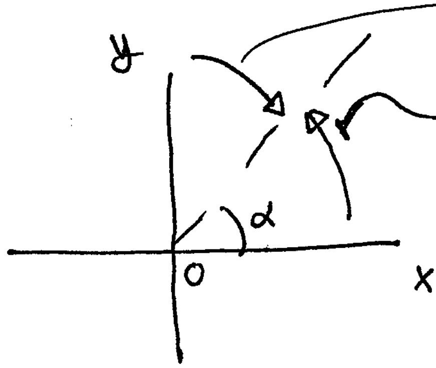
If we let α denote any real number and restrict the value of θ in expression so that $\alpha < \theta < \alpha + 2\pi$ the function

$$\log z = \ln r + i\theta \qquad (r>0,\ \alpha < \theta < \alpha + 2\pi)$$

with components

$$u(r,\theta) = \ln r \quad \text{and} \quad v(r,\theta) = \theta$$

is single valued and continuous. If we define on the ray $\theta = \alpha$ it would not be continuous



Approach this way:

$\lim v(r,\theta) = \alpha + 2\pi$

$\lim v(r,\theta) = \alpha$

Thus, not continuous.

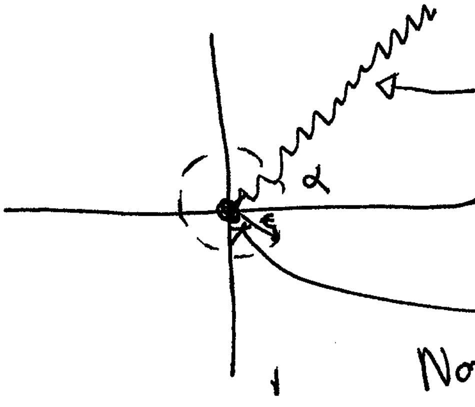
Furthermore, can show that C.R. equations are satisfied

Polar form: $r u_r = v_\theta \quad u_\theta = -r v_r$

Thus, is analytic for this choice

So we can define Single Valued Function $\operatorname{Log} z$ by restricting to $\alpha < \theta < \alpha + 2\pi$

$$\operatorname{Log} z = \log z \qquad (r>0,\ \alpha < \theta < \alpha + 2\pi)$$



Function not defined on this line "branch cut" → Points on branch cuts

neighboorhood intersects branch cut

origin of branch cut is called "branch point"

Notice it is not an isolated singularity since every

Residues

Recall that for any function ~~analytic~~ $f(z)$ with isolated singularity at z_0, it is analytic in the domain $0 < |z - z_0| < R_2$ with

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \cdots$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} \qquad (n = 1, 2, \ldots)$$

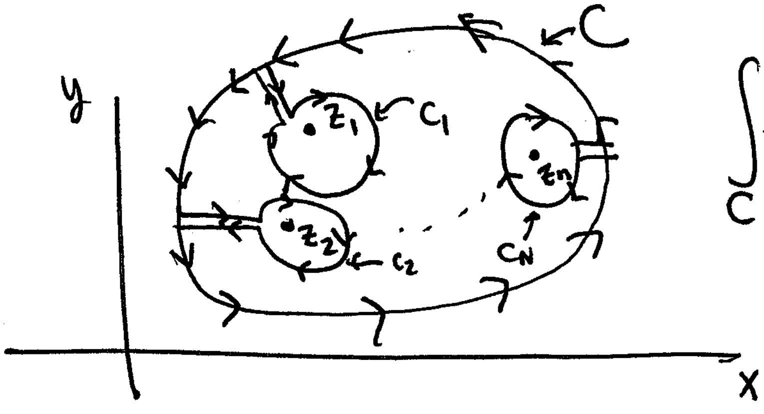
So we know

$$\int_C f(z)\,dz = 2\pi i\, b_1$$

The complex number b_1, which is the coefficient of $\frac{1}{z - z_0}$ is called the <u>residue of f at z_0</u>.

We already saw this earlier.

Now consider the case where there are ~~not~~ finite number of isolated singularities z_K ($K = 1, 2, \ldots n$) inside a Contour C



Then from picture we know

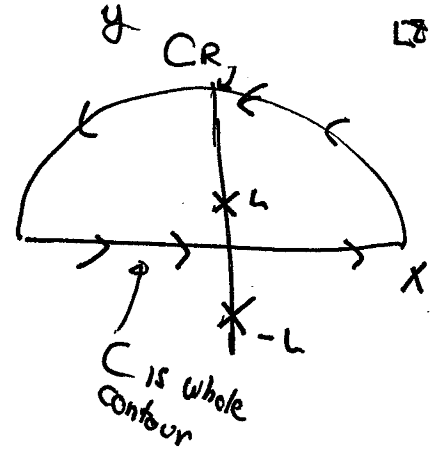
$$\int_C f(z) = \sum_{J=1}^{N} \int_{C_J} f(z)\,dz \quad \Bigg\} \text{(usual argument) from picture on left}$$

$$= \sum_{J=1}^{N} 2\pi i \operatorname*{Res}_{z = z_K} f(z)$$

Example

Consider integral

$$\frac{\pi}{2} = \int_0^{\infty} \frac{dx}{x^2+1} = \frac{1}{2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx}_{\text{call this } I}$$



So we have

$$I + \int_{C_R} \frac{1}{z^2+1}\,dz = \int_C \frac{dz}{z^2+1}$$

$$I = \int_C \frac{dz}{z^2+1} - \int_{C_R} \frac{1}{z^2+1}\,dz$$

$$= \int_C \frac{dz}{z^2+1} - \int_{C_R} \frac{1}{z^2+1}\,dz$$

To see is zero

notice that when $|z| = R$

$$|f(z)| \leq \frac{1}{R^2}$$

So

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{\pi R}{R^2}$$

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{\pi}{R}$$

$\rightarrow 0$ as $R \to \infty$

$$= 2\pi i \underbrace{\operatorname{Res}(i)}_{} - 0$$

So now calculate residue at $z = +i$

$$\frac{1}{z^2+1} = \frac{1}{z+i}\left(\frac{1}{z-i}\right)$$

$$= \frac{1}{z+i} \approx \underbrace{\left.\frac{1}{z+i}\right|_{z=i} - \left.\left(\frac{1}{z+i}\right)^2\right|_{z=i}(z-i) + \cdots}_{\text{Taylor expand}}$$

$$= \frac{1}{2i} - \frac{1}{(2i)^2}(z-i) + \cdots$$

So residue is

$$\frac{1}{2i} \Rightarrow I = \pi \Rightarrow \int_0^{\infty} \frac{dx}{1+x^2} = \frac{1}{2}I = \frac{\pi}{2}$$

Residue at Poles + Residues at infinity

Theorem An isolate point z_0 of a function f is a "pole" of "order m" iff

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

where $\phi(z)$ is analytic and non-zero at z_0.

switch order

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}} \quad \text{where } \phi^{(0)}(z_0) = \phi(z_0)$$

Definition ~~Meaning~~ $f(z)$ has a pole of order m at z_0 if the Laurent series of $f(z)$ is on the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

Proof Write $f(z)$ in Taylor series

$$f(z) = \sum_{j=0}^{\infty} \frac{\phi^{(j)}(z_0)}{j!} \frac{(z-z_0)^j}{(z-z_0)^m} = \sum_{j=0}^{\infty} \frac{\phi^{(j)}(z_0)}{j!} (z-z_0)^{j-m}$$

$\operatorname*{Res}_{z=z_0} f(z)$ is coefficient on $\frac{1}{z-z_0}$ term, in this case

$$= \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

Converse follows similarly.

<u>Example 1</u> Consider two functions $p(z)$ and $q(z)$ which are analytic at z_0. If $p(z_0) \neq 0$, $q(z_0) = 0$ and $q'(z_0) \neq 0$

then
$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

because
$$\frac{p(z)}{q(z)} = \frac{p(z)}{(z-z_0)g(z)} = \frac{\phi(z)}{(z-z_0)}$$

where $\phi(z) = \dfrac{p(z)}{g(z)}$

Since this implies
$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \phi(z_0) = \frac{p(z_0)}{g(z_0)} = \frac{p(z_0)}{q'(z_0)}$$

Here we have used $g(z_0) = q'(z_0)$ since
$$q(z) = (z-z_0)q'(z_0) + (z-z_0)^2 \frac{q^{(2)}(z_0)}{2!} + \cdots$$
$$= (z-z_0)\left[g(z)\right]$$
So all higher derivatives terms are zero

<u>Example 2</u>

$$f(z) = \cot z = \frac{\cos z}{\sin z}$$

Singularities at $z = n\pi$ $n \in \mathbb{Z}$ $(n = 0, \pm 1, \pm 2, \cdots$

notice $\frac{d}{dz} \sin z = \cos z$ so that example above applies

$$\boxed{\operatorname*{Res}_{z=n\pi} \cot z = \frac{\cos n\pi}{\cos n\pi} = 1}$$