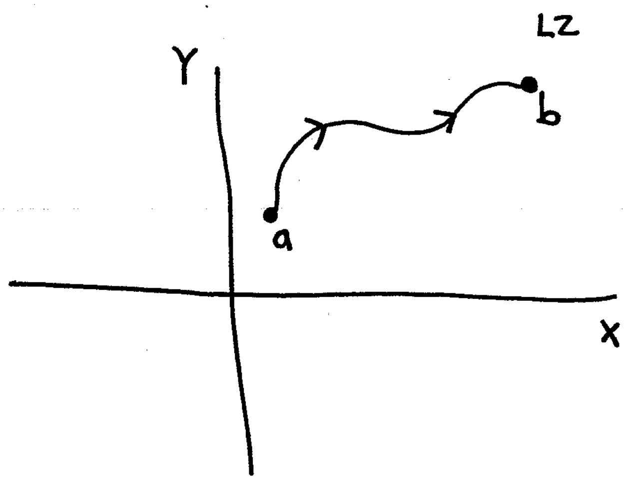
V Integration

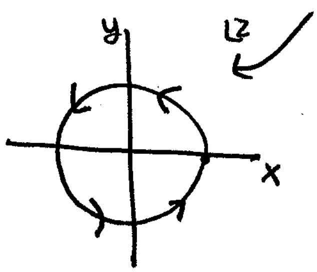
Consider "Contour" (curve) C in z-plane. Integrals of complex variables are "line integrals". They are defined on curves C instead of just intervals on the real line.

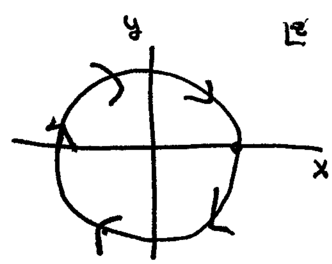


Curves have both line + direction

Example

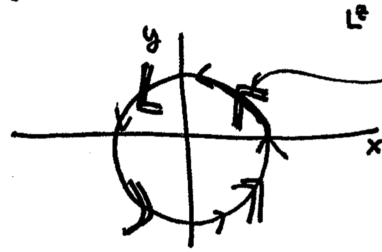
Curve $z = e^{i\theta}$ $0 \le \theta \le 2\pi$ is a unit circle around origin





This is not the same as $z = e^{-i\theta}$ $0 \le \theta \le 2\pi$ (Set of points is same but now direction changes)

Finally $z = e^{2i\theta}$ $0 \le \theta \le 2\pi$ has same set of points as above, but is transversed twice



go around second time.

Defining contour integral of function $f(z)$ along contour C

$$\int_C f(z)\,dz$$

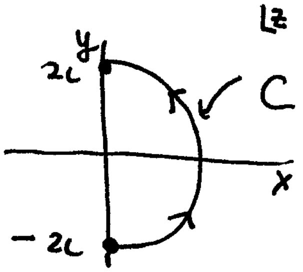
Two ways to think about it.

① Suppose that we can respresent the contour by $z = z(t)$ $(a \le t \le b)$ extending from $z_1 = z(a)$ to $z_2 = z(b)$. Assume $f(z(t))$ is piecewise continous of C. We can define the <u>contour integral</u> of f along C

as
$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt$$

Example

Let C be given by $z(t) = 2e^{it}$ $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ and let $f(z) = \bar{z}$



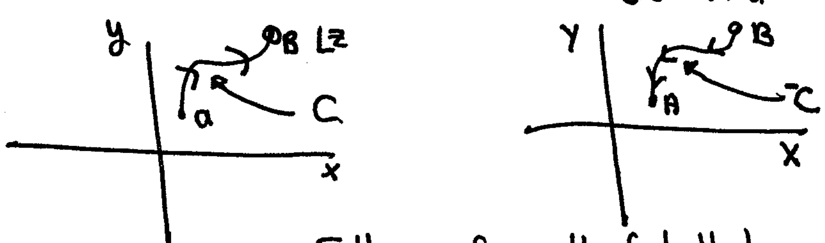
$$\frac{d}{dt}(2e^{it}) = 2e^{it}\,i$$

$$I = \int_C f(z)\,dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overline{2e^{it}}\,(2e^{it})\,i\,dt$$

$$= 4i\int_{-\pi/2}^{\pi/2} dt = 4\pi i$$

Example 2

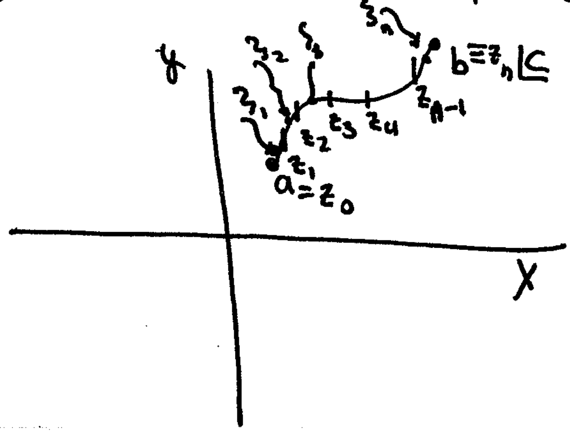
Denote $-C$ the contour obtained by C be reversing directions



Then
$$\int_C f(z)\,z = -\int_{-C} f(z)\,dz$$

Follows from the fact that $-C$ if given by $z = z(-t)$ $(-b \le t \le -a)$

② Can also think of this as Reimann sum



Divide curves into n intervals.

Define sum

$$S_n = \sum_{j=1}^{n} f(\xi_j)(z_j - z_{j-1})$$

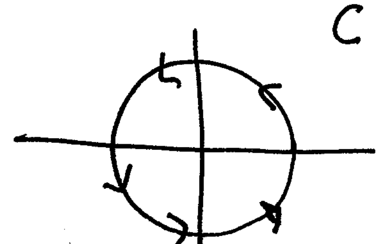
Let $n \to \infty$ with $|z_j - z_{j-1}| \to 0$. If the limit $\lim_{n\to\infty} S_n$ exists independen of details choosing points $\{z_i, \xi_i\}$, then

$$\int_C f(z)dz = \lim_{n\to\infty} S_n.$$

Example

$$I = \int_C z^{a-1} dz$$

C



C is defined by $z(\theta) = Re^{i\theta}$ (notationally convenient to use θ instead of t)

$$f(z(\theta))\, z'(\theta) = iR^a e^{ia\theta} = -R^a \sin a\theta + iR^a \cos a\theta$$

$$I = iR^a \int_{-\pi}^{\pi} e^{ia\theta} d\theta = iR^a \left[\frac{e^{ia\theta}}{ia}\right]_{-\pi}^{\pi} = \frac{i2R^a}{a}\left(\frac{e^{ia\pi} - e^{-ia\pi}}{2i}\right)$$

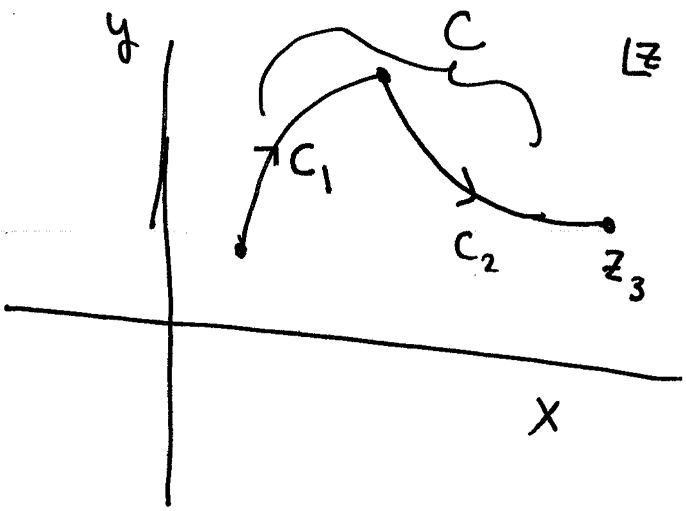
$$= \frac{i2R^a}{a} \sin a\pi$$

IF $a = 0$ $\boxed{I = 2\pi i}$

Thus, if a is non-zero integer ($a = \pm 1, \pm 2, \cdots$) $I = 0$

The last example is an important one. Make sure you understand it well!

Some properties of contour integrals



Let C be piece-wise continous curve formed by summing two curves C_1 and C_2

Then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

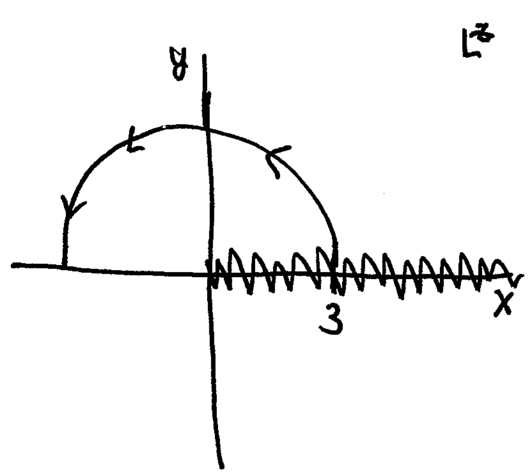
$$\int_C [f(z)+g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$$

$$\int_C [z_0 f(z)]\,dz = z_0 \int_C f(z)\,dz$$

Final example

Let C denote $\cancel{z_0} = 3e^{i\theta}$ $\quad (0 \le \theta \le \pi)$

$f(z) = z^{1/2}$ $\qquad I = \int_C f(z)\,dz$



Note $z^{1/2}$ is a multivalued function. To make it single-valued we introduce a "branch-cut" along positive axis. The function $z^{1/2}$ is not defined here.

Nonetheless, the $\lim_{z\to 3} f(z) = \lim_{\theta\to 0} 3e^{i\theta}$ does exist and equals 3. Thus we can still perform integral

$$f(z(\theta))\,z'(\theta) = \sqrt{3}\,e^{i\theta/2}\,3ie^{i\theta} = 3\sqrt{3}i\,e^{i3\theta/2}$$

$$I = \int_C f(z)\,dz = 3\sqrt{3}i\int_0^{\pi} e^{i3\theta/2}\,d\theta = \left[\frac{-2}{3i}(1+i)\right]3\sqrt{3}i$$

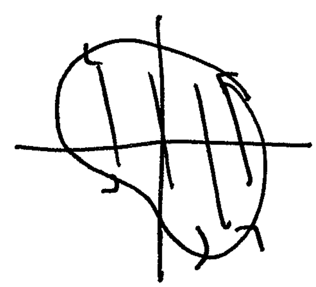
$$= -2\sqrt{3}\,(1+i)$$

VI ~~Cauchy Integral Theorem~~ Cauchy-Goursat Theorem

Theorem If a function is analytic at all points interior to and on simple contour C, then (ie the region inside C has no "holes")

$$\int_C f(z)\,dz = 0$$

Proof 1



$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+idy)$$

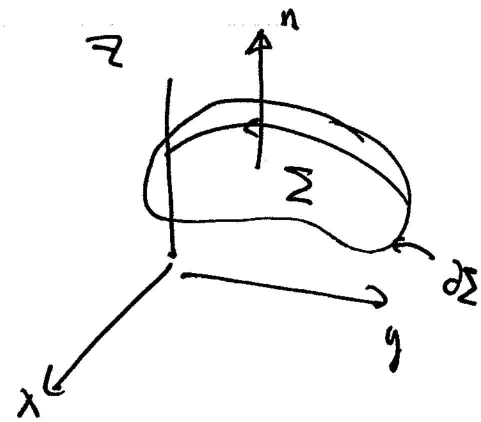
$$= \oint_C (u\,dx - v\,dy) + i\oint (v\,dx + u\,dy)$$

STOKES THEOREM / Green's Theorem — Aside

For a vector $\vec{A} = \hat{x} A_x + \hat{y} A_y$

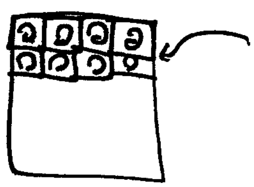
$$\oint_c (A_x\, dx + A_y\, dy) = \int \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dx\, dy \qquad \text{(Green's Theorem)}$$

More generally $\vec{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{z}$



$$\int_{\Sigma} \nabla \times \vec{F}\, d\vec{S} = \oint_{\partial\Sigma} F \cdot dr$$

Look at proof. Break surface into litle square



notice interior lines cancel. So have to show for small squares thing hold

"simplices"
Basis for all of Algebraic topology

So then this implies

$$\oint (u\,dx - v\,dy) = -\int \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dx\, dy$$

$$\oint (v\,dx + u\,dy) = \int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\, dy$$

$$\oint f(z)\, dz = -\int \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dx\, dy + i \int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\, dy = 0$$

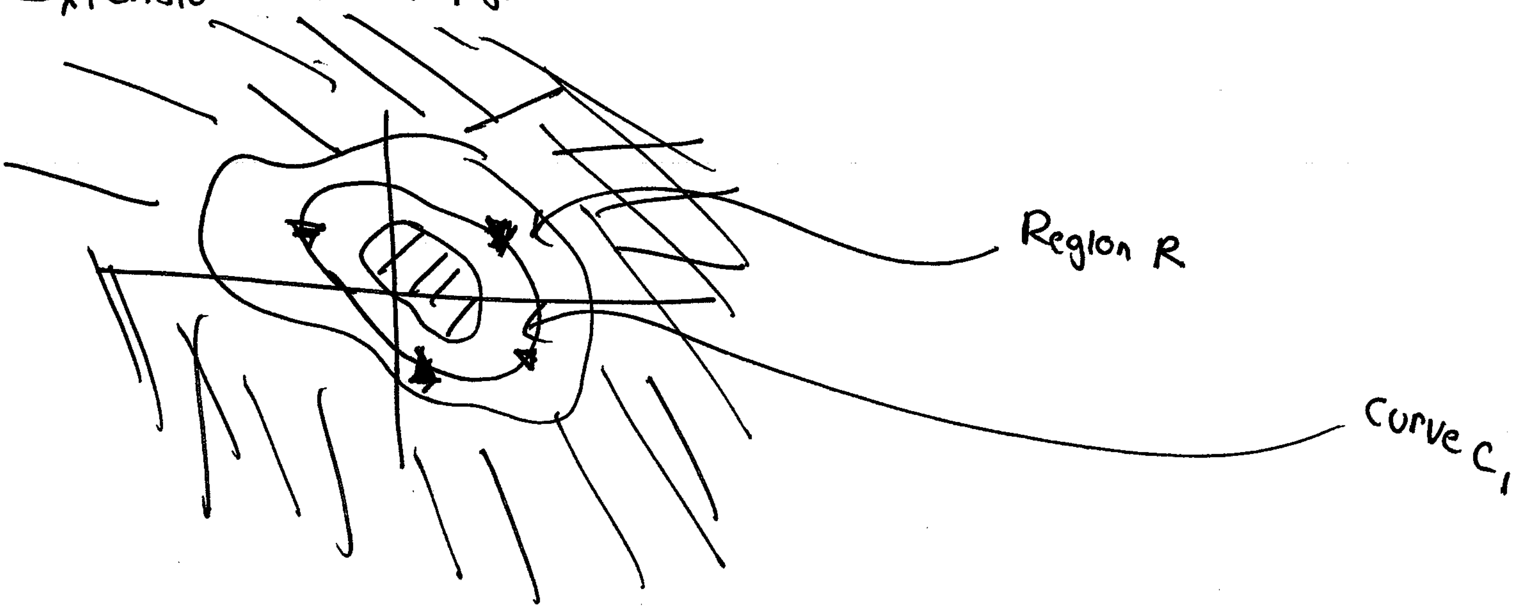
By C.R. Equations

Proof 2 Use limits and definition of derivatives (Goursat Proof)

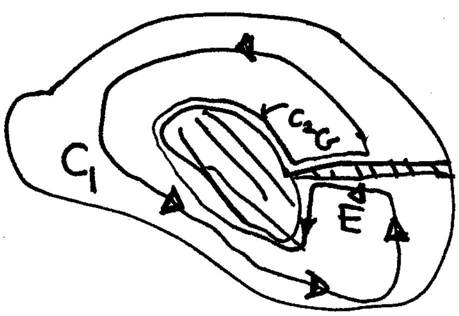
See Churchill Brown, Arken 6.3, etc.

There are lots of trick with how to "choose" contours.

Extension to multiply connected region:



Can convert this into "simply-connected region"



$$\oint_{C_1} f(z) + \oint_{C_2} f(z) + \int_E f(z) + \int_G f(z) = 0$$

In general this will true of any two curve in multiply connected region

Since $\oint_E f(z) = -\oint_G f(z)$ (In limit where sliver goes to zero / same curve opposite directions)

$$\boxed{\oint_{C_1} f(z) + \oint_{-C_2} f(z) = 0} \Longrightarrow \boxed{\oint_{C_1} f(z) = \oint_{C_2} f(z)}$$

VI Cauch Integral Theorem (Formula)

We are now in the position to prove Cauchy's integral theorem.

Theorem Let f be analytic everwhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then

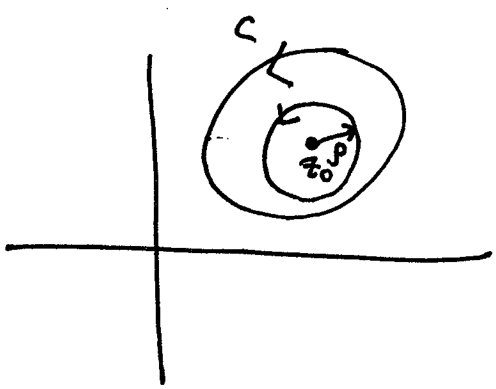
$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)}$$

Example Let C be positively oriented circle $|z|=2$. Since the function $f(z) = \frac{z}{9-z^2}$ is analytic within and on C.

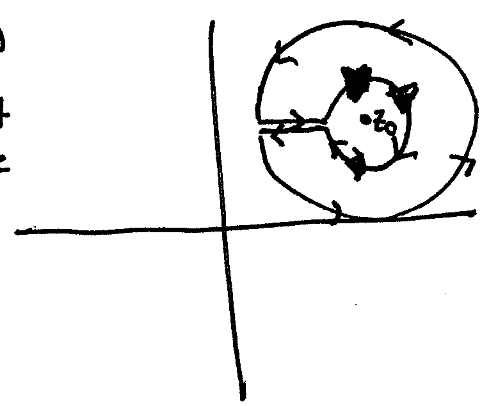
We can calculate

$$\oint_C dz \, \frac{z\,dz}{(9-z^2)(z+i)} = \int_C \frac{z/9-z^2}{z-(-i)} = 2\pi i \left(\frac{-i}{10}\right) = \frac{\pi}{5}$$

Proof



By deforming as in last example $\Longrightarrow$



We can show

$$\int_C \frac{f(z)\,dz}{z-z_0} = \int_{C_\rho} \frac{f(z)}{z-z_0}$$

So we can write

$$\oint_C \frac{f(z)\,dz}{z-z_0} - f(z_0)\underbrace{\oint_{C_\rho} \frac{dz}{(z-z_0)}} = \oint_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}$$

$$\oint_{C_\rho} \frac{dz}{z-z_0} = 2\pi i$$

So this becomes

$$\int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i\, f(z_0) = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}$$

Since f is analytic. We show that we shrink $\rho \to 0$ that the right hand side goes to zero. Well let $z = z_0 + re^{i\theta}$

$$\underbrace{\lim_{r\to 0} \int_{C_\rho} \frac{f(re^{i\theta}+z_0)-f(z_0)}{re^{i\theta}} \approx \oint_{C_\rho} f'(z)}_{\text{Can make this more rigorous using } \varepsilon,\ \delta\text{'s (see Churchill Brown)}} \doteq 1$$

Since contour is closed!

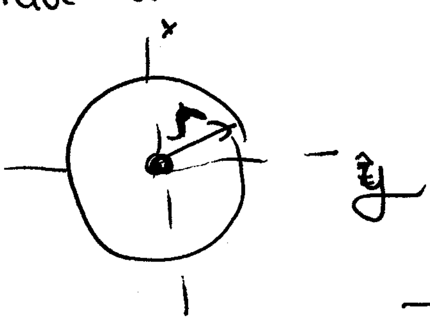
We have proved something incredible!

The value of an analytic function $f(z)$ is given at an interior point $z = z_0$ once we know the value on the surface.

EM + Gauss's Law

Analogy from E.M. with 2-Dimensional Gauss's law

If we have a line of charge in z-direction. Once I tell you integral of $\int E \cdot ds$ on surface, you can tell be value of E anywhere.



Consider infinite-cylinder axis along z-direction.
This, by symmetry, is like line-integral in x-y plane $\Longrightarrow$ related to line charge.

$$\varphi = -\frac{q \ln \rho}{2\pi \epsilon_0} \qquad E = -\nabla\varphi = \underbrace{\frac{q\hat{\rho}}{2\pi\epsilon_0 \rho}}_{\text{looks like } \frac{1}{r}}$$

$$r = \frac{1}{z - z_0}.$$

C.R. Equations tell us $\nabla \times \vec{E} = 0$ (So we can really think of E.F. analogy!)

"Analatycity" corresponds to having from C.R.

~~no additional charges around!!~~

Extension

$$\frac{d^n f(z_0)}{dz_0^n} = \frac{1}{2\pi i}\int_C \frac{d^n}{dz_0^n}\frac{f(z)}{(z-z_0)} = \frac{1}{2\pi i}\oint_C f(z)\,\frac{d}{dz^n}\frac{1}{(z-z_0)}$$

$$= \frac{1}{2\pi i}\int_C f(z)\left[\frac{n!}{(z-z_0)^{n+1}}\right]$$

$$\boxed{f^{n}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}}$$

This can be made more rigorrous and switching integral and derivative can be justified.

Morera's Theorem If a function $f(z)$ is continuous in a simply connected region R and $\oint_C f(z)dz = 0$ for every closed contour within C, then $f(z)$ is analytic throughout R.

Example If C is positively oriented unit circle $|z|=1$ and

$$f(z) = \exp(2z)$$

then

$$\int_C \frac{\exp(2z)}{z^4} = \int_C \frac{f(z)dz}{(z-0)^{3+1}} = \frac{2\pi i}{3!} f'''(0) = \frac{8\pi i}{3}$$