Complex Analysis

We will spend the next few ~~wo~~ weeks reviewing and learning complex analysis. You can read Chapter 6 and 7 of Arken. Another good resource is Brown + Churchill.

I Complex Numbers

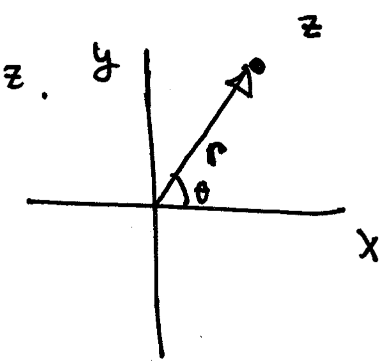
A complex number z is an ordered pair of real numbers (x, y) usually written

$$z = x + iy \quad \text{(rectangular coordinates)} \qquad i = \sqrt{-1}.$$
$$= re^{i\theta} \quad \text{(polar coordinates)}$$

$x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ are the real and imaginary parts of z.

The quantity r is the modulus or absolute value of z.

$$r = \sqrt{x^2 + y^2} = |z|$$



The argument of z is

$$\arg(z) = \theta = \tan^{-1} y/x \quad \text{(with appropriate attention to quadrant)}$$

Notice

$$e^{i\theta} = e^{i(\theta + 2\pi n)} \quad \text{where } n \text{ is an integer.}$$

The principal value of $\arg z$ is in the range

$$-\pi < z < \pi$$

Identities to know:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

For $n > 1$

$$e^{i\theta n} = \cos n\theta + i\sin\theta n$$

The "Complex Numbers" are a field. There are a number of operations

$$z_1 = x_1 + i y_1 \qquad z_2 = x_2 + i y_2$$
$$\quad = r_1 e^{i\theta_1} \qquad\quad = r_2 e^{i\theta_2}$$

Addition $\quad z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) = (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r\sin\theta_1 + r\sin\theta_2)$

Multiplication $\quad z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$

$$= r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

Complex Conjugation

$$\bar{z}_1 = (x_1 - i y_1) = r_1 e^{-i\theta_1}$$

Also notice

$$z\bar{z} = x^2 + y^2 = |z|^2 = r^2$$

Triangle Inequality

$$|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

n-th roots of a complex number:

$$z^{1/n} = (re^{i\theta})^{\frac{1}{n}}$$

From Fundamental Theorem of Algebra there are n roots. Call them

C_k [$k = 0, 1, 2, \dots n-1$]

Easy to check

$$C_k = \sqrt[n]{r}\exp\left[i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]$$

are roots because $e^{i(\theta + 2\pi j)} = e^{i\theta}$ for all integers j.

II Complex Functions of a complex Variable

We can define function of complex variables that assigns a ~~Let S be a set of complex Numbers~~ complex number w to each z

$$w = f(z)$$

In terms of real numbers. Let us write

$$z = x + iy \quad \text{and} \quad w = u + iv.$$

Then

$$f(z) = u(x,y) + iv(x,y).$$

If we use polar coordinates r and θ instead of x and y then

$$u + iv = f(re^{i\theta}) \quad \text{and we can write}$$

$$f(z) = u(r,\theta) + iv(r,\theta).$$

<u>Example</u> $\quad f(z) = z^2$

$$f(x+iy) = x^2 - y^2 + 2ixy$$

$$u(x,y) = x^2 - y^2 \quad \text{and} \quad v(x,y) = 2xy$$

For Polar coordinates

$$f(re^{i\theta}) = r^2\cos 2\theta + ir^2\sin 2\theta$$

$$u(r,\theta) = r^2\cos 2\theta \qquad v(r,\theta) = r^2\sin 2\theta$$

Important thing about a function is that it is <u>single-valued</u>. It assigns a single value w to each z.

We can <u>generalize</u> this <u>multi-valued</u> functions where a function assigns more than one value to each z. In this case, we must choose just <u>one</u> of the assignments to construct a single valued function from the multivalued function.

Example Conside "square root) For $z = re^{i\theta}$

$$f(z) = z^{\frac{1}{2}} = \pm\sqrt{r}\,\exp\left(\frac{i\theta}{2}\right) \quad \text{and} \quad \underset{\text{of arg } z}{-\pi < \theta \le \pi} \text{ is P.V}$$

So in general this assigns two complex numbers to each z.

To make S.V. (single-valued) choose just positive branch

$$f(z) = \sqrt{r}\,\exp\left(\frac{i\theta}{2}\right) \qquad r > 0,\ -\pi < \theta \le \pi$$

With the additional assignment

$$f(0) = 0$$

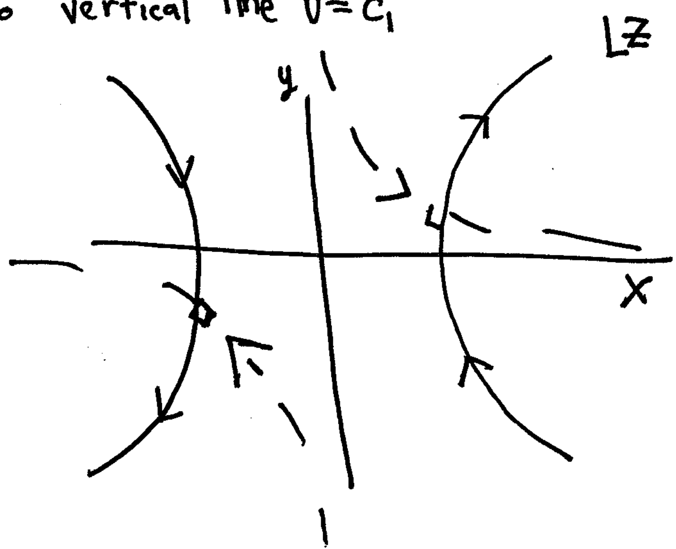
This is defined on whole plane.

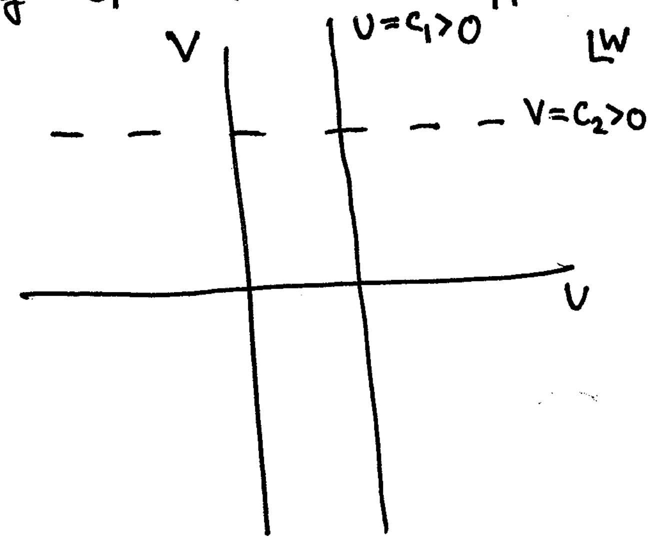
$\boxed{\text{Think about logarithm function!}}$

We can visualize a function $f(z)$ of the complex plane z to the plane W by looking at what the function does to a family of points. Consider again

$$W = z^2 \Longrightarrow \quad U = (x^2 - y^2) \qquad V = 2xy.$$

So each branch of a hyperbola $x^2 - y^2 = C_1$ $(C_1 > 0)$ is mapped unto vertical line $U = C_1$

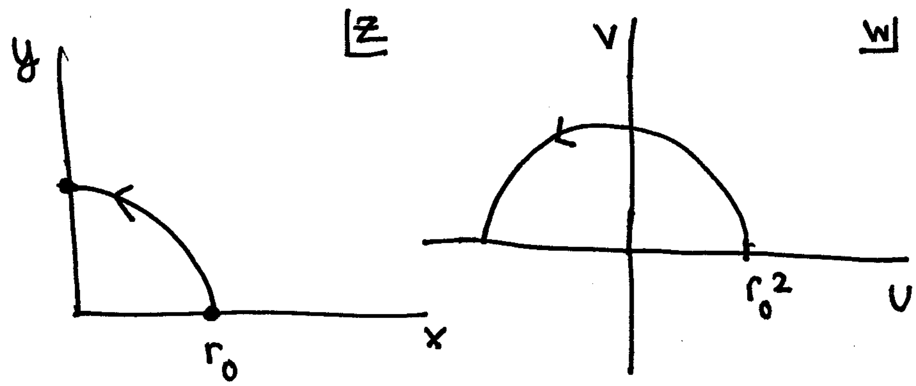




Consider agan $f(z)=z^2$. We can ask what does this function due to a certain region (ie. what is the "image" of a region.

Consider this mapping in polar form. $z = re^{i\theta}$ $w = \rho e^{i\phi}$

$$\omega = r^2 e^{i2\theta} \to \rho = r^2 \quad \phi = 2\theta$$



So maps quater circle in first quadrant to half-circle. Thus, it is clear this maps the first quadrant $r \geq 0,\ 0 \leq \theta \leq \frac{\pi}{2}$ to upper-half plane $\rho \geq 0,\ 0 \leq \phi \leq \pi$.

Common Complex Functions

Most of the common ~~complex~~ real valued trigonometric, exponential and hyperbolic functions can be extended to the complex plane using analytic continuation.

For example

$$e^z = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!}$$

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{n!} = \frac{e^{iz} - e^{-iz}}{2i}$$

etc.

Most of the usual formulas you are used to still hold

See Chapter 3 of Churchill+Brown for details + Excercises Arken 6.1.

What does analytic continuation mean?

Example

Conside a plan wave of light of angular frequency ω

This is represented by

$$e^{i\omega(t-\frac{nx}{c})}$$

Where n is the index of refraction. Most of the time n is real. However, we con replance n by a "complex index of refraction" $n-ik$. $\Longrightarrow$ So expression becomes

$$\propto e^{i\omega(t-\frac{nx}{c})}\, e^{-\frac{\omega k x}{c}}$$

Physically, this means that wave "attenuates" $\rightarrow$ scattering dampens wave. Occurs a lot $\rightarrow$ Include "dissapation" by making things complex.

III Limits, Derivatives, and all that

The limit

$$\lim_{z\to z_0} f(z) = w_0$$

means that for each positive number ε, there is a positive number δ s.t. $|f(z)-w_0|<\varepsilon$ whenever $0<|z-z_0|<\delta$

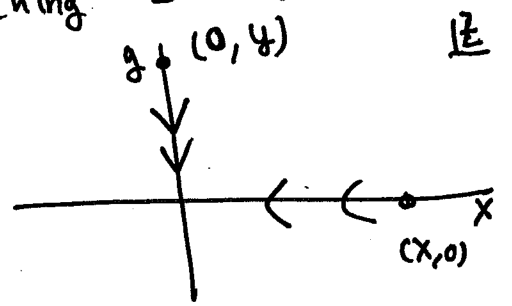
In words, it means that as you approach z_0 from **any** direction, $f(z_0) \to w_0$. It is independent of path towards z_0 in complex plane

Example Consider.

$$f(z) = \frac{z}{\bar{z}}$$

The limit $\lim\limits_{z\to 0} f(z)$ does **not** exist

Consider approaching zero from positive x-axis and positive y-axis



For $z = (x,0) = x + i0$

$$f(z) = \frac{z}{\bar{z}} = \frac{x+i0}{x+i0} = 1$$

For $z = (0,y) = 0 + iy$

$$f(z) = \frac{0+iy}{0-iy} = -1 ,$$

Thus as $z \to 0$ along positive x axis $f(z)$ approaches 1 but along y axis $f(z)$ approaches z -1 So does not exist.

Theorems on limits

Theorem 1

Suppose that

$$f(z) = u(x,y) + i v(x,y) \qquad z = x + iy$$

and

$$z_0 = x_0 + iy_0 \qquad w_0 = u_0 + i v_0$$

Then

$$\lim_{z \to z_0} f(z) = w$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

Proof See any complex analysis's textbook

Theorem 2 Suppose that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} F(z) = W_0$$

then

(A) $$\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0$$

(B) $$\lim_{z \to z_0} [f(z) F(z)] = w_0 W_0$$

(C) if $W_0 \neq 0$

$$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}.$$

We are now in position for the following definition.

Def A function is continuos if all three of the following conditions are met:

i) $\lim_{z \to z_0} f(z)$ exists

ii) $f(z_0)$ exists

iii) $\lim_{z \to z_0} f(z) = f(z_0)$

We can define the derivative of a function in a neiborhood of z_0 as:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}$$

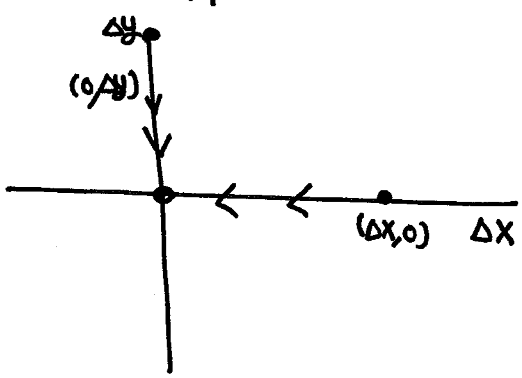
This is usual definition of derivative in calculus. Remember though, limit must <u>exist</u> no matter how you approach z_0

Example

Consider $W = f(z) = |z|^2 = z\bar{z}$

$$\lim_{\Delta z \to 0} \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \bar{z} + \underbrace{\overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z}}_{A}$$

Now again approach in two directions positive x-axis, negative y-axis



when $\Delta z = (\Delta x, 0)$

$$\lim_{\Delta x \to 0} A^{\Delta x} = \bar{z} + \Delta z + z = \bar{z} + z$$

when $\Delta z = (0, \Delta y)$

$$\lim_{\Delta z \to 0} A^{\Delta y} = \bar{z} - \Delta z - z = \bar{z} - z$$

These are not equal $A^{\Delta x} \neq A^{\Delta y}$ unless $z = 0$.

Thus, $\frac{dw}{dz}$ <u>does not</u> exist except at $z = 0$.

All usual formulas apply. For c a complex constant:

$$\frac{dc}{dz}=0 \qquad \frac{d}{dz}z=1 \qquad \frac{d}{dz}[cf(z)]=cf'(z)$$

$$\frac{d}{dz}z^n = nz^{n-1}$$

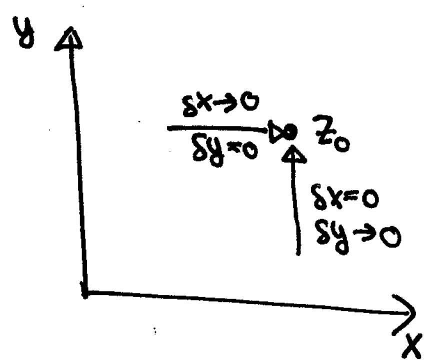
$$\frac{d}{dz}[f(z)+g(z)] = f'(z)+g'(z)$$

$$\frac{d}{dz}[f(z)g(z)] = f(z)g'(z)+f'(z)g(z) \qquad \text{[Product rule]}$$

IV Cauchy-Riemann Equations

Write

$$f(z) = u(x,y) + iv(x,y)$$



Consider taking limit in two different ways

$$\delta z = \delta x + i\delta y$$
$$\delta f = \delta u + i\delta v$$

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}$$

If we approach along x axis

$$\lim_{\delta z\to 0}\frac{\delta f}{\delta z} = \lim_{\delta x\to 0}\left(\frac{\delta u}{\delta x}+i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\lim_{\delta z\to 0}\frac{\delta f}{\delta z} = \lim_{\delta y\to 0}\left(-i\frac{\delta u}{\delta y}+\frac{\delta v}{\delta y}\right) = -i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}.$$

Thus, for derivatives to exist at z_0 need

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

These are the famous Cauchy-Riemann Conditions for a derivative to exsist. $\frac{df}{dz}$

In fact one can show an even stronger statement using Theorem 1 on limits. If the Cauchy-Riemann conditions are satisfied at a point. $z_0 = x_0 + iy_0$, then the derivative $f'(z_0)$ exists.

Thus, $f'(z_0)$ exists iff the C.R. conditions are satisfied.

Excercise: Derive in Polar coordinates!

Important Implication

The curves $u = c_1$ will always be orthognal to the curves $v = c_2$. Thus, any function $f(z)$ whose derivate exists maps to an "orthognal" coordinate system."

Tangent to $u(x, y)$ has slope $-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \equiv -\frac{u_x}{u_y}$

Tangent to $v(x, y)$ has slope $-\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} \equiv -\frac{v_x}{v_y}$

C.R. Equations imply $\frac{u_x}{u_y} \frac{v_x}{v_y} = -1$. (i.e. They are orthognal)

A function $f(z)$ is <u>analytic</u> at some point z_0 if it has a derivative at each point in some neighborhood of z_0.

One also use terms <u>regular</u> or <u>holomorphic</u>.

A function is <u>entire</u> if it is analytic everywhere.

<u>Example 1</u>

$$f(z) = \frac{1}{z^2+3}$$

This is analytic everywhere except $z = \pm\sqrt{3}\,i$

Since $f'(z) = \dfrac{-2z}{(z^2+3)}$. This exists everywhere except

<u>Example 2</u> Harmonic Functions

Let $H(x,y)$ be some function that satisfies "Laplace's Equation"

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0.$$

Such a function is called Harmonic.

Any function $f(z) = u(x,y) + i v(x,y)$ is analytic. Then $u(x,y)$ and $v(x,y)$ are Harmonic.

Proof: $U_x = V_y \qquad U_y = -V_x \qquad$ (C.R equations)

$U_{xx} = V_{yx} \qquad U_{yx} = -V_{xx} \qquad$ Dif. w.r.t x

$U_{xy} = V_{yy} \qquad U_{yy} = -V_{xy} \qquad$ Dif w.r.t y

Since partial derivatives commute get $U_{xx} + U_{yy} = 0$ and $V_{xx} + V_{yy} = 0$.