1 ALGEBRAIC PRELIMINARIES

It is somewhat unusual to begin a physics textbook with algebraic identities, which are in general hidden in appendices. However, our discussion of perturbative aspects of quantum mechanics and quantum field theory will entirely be based on path or functional integrals and more generally functional techniques. Therefore a reader not already familiar with these concepts may find it difficult to follow the algebraic manipulations which enter in the derivation of many results. Moreover we want to indicate by such a choice that the various technical difficulties which we shall meet, will in general be directly confronted rather than carefully hidden.

Therefore in this first chapter we recall a few algebraic identities about gaussian integrals. We also recall the concept of functional differentiation and the algebraic definition of the determinant of an operator.

We then define and discuss a few properties of differentiation and integration in a Grassmann, i.e. antisymmetric algebra. In particular we calculate gaussian "fermionic" integrals. Throughout the chapter all expressions are given for a finite but arbitrary number of variables, because the focus is mainly on algebraic properties. However the generalization to an infinite number of variables will be easy, as will be discussed in the following chapters.

Note that in this chapter, as well as in this whole work, *summation over repeated indices will always be meant* (except if explicitly stated otherwise).

1.1 The Gaussian Integral

In this section we briefly review a few algebraic properties of gaussian integrals in the case of a finite number of integration variables.

A general gaussian integral has the form:

$$I(\mathbf{A},\mathbf{b}) = \int \left(\prod_{i=1}^{n} \mathrm{d}x_i\right) \exp\left(-\sum_{i,j=1}^{n} \tfrac{1}{2} x_i A_{ij} x_j + \sum_{i=1}^{n} b_i x_i\right), \tag{1.1}$$

in which $\mathbf{A}$ is a symmetric matrix with eigenvalues λ_i satisfying

$$\mathrm{Re}(\lambda_i) \ge 0\,, \quad \lambda_i \ne 0\,.$$

To calculate I one first looks for the minimum of the quadratic form:

$$\frac{\mathrm{d}}{\mathrm{d}x_k}\left(\sum_{i,j=1}^{n} \tfrac{1}{2} x_i A_{ij} x_j - \sum_{i=1}^{n} b_i x_i\right) = 0\,.$$

The solution is:

$$x_i = \left(A^{-1}\right)_{ij} b_j\,, \tag{1.2}$$

(summation over j being meant as stated above) and one sets:

$$x_i = \left(A^{-1}\right)_{ij} b_j + y_i\,. \tag{1.3}$$

The integral becomes:

$$I = \exp\left[\tfrac{1}{2}b_i\left(A^{-1}\right)_{ij}b_j\right]\int\left(\prod_i \mathrm{d}y_i\right)\exp\left(-\tfrac{1}{2}y_iA_{ij}y_j\right). \tag{1.4}$$

The last integral can be calculated by changing variables, setting

$$\left(A^{1/2}\right)_{ij}y_j = y'_i\,,$$

the eigenvalues $\lambda_i^{1/2}$ of $\mathbf{A}^{1/2}$ being chosen such that $-\pi/4 \le \operatorname{Arg}\lambda_i^{1/2} \le +\pi/4$. The integral over the y'_i's is then straightforward and one obtains:

$$I(\mathbf{A},\mathbf{b}) = (2\pi)^{n/2}(\det\mathbf{A})^{-1/2}\exp\left[\sum_{i,j=1}^{n}\tfrac{1}{2}b_i\left(A^{-1}\right)_{ij}b_j\right]. \tag{1.5}$$

By differentiating this last expression with respect to the variables b_i, it is then possible to calculate the average of any polynomial with a gaussian weight:

$$\langle x_{k_1}x_{k_2}\dots x_{k_\ell}\rangle = \mathcal{N}\int\left(\prod_i \mathrm{d}x_i\right)x_{k_1}x_{k_2}\dots x_{k_\ell}\exp\left(-\sum_{i,j=1}^{n}\tfrac{1}{2}x_iA_{ij}x_j\right), \tag{1.6}$$

in which the normalization $\mathcal{N}$ is chosen in such a way that $\langle 1\rangle = 1$:

$$\mathcal{N}^{-1} = I(\mathbf{A},0).$$

Indeed from expression (1.1) one derives:

$$\frac{\partial}{\partial b_k}I(\mathbf{A},\mathbf{b}) = \int\left(\prod_i \mathrm{d}x_i\right)x_k\exp\left(-\tfrac{1}{2}x_iA_{ij}x_j + b_ix_i\right). \tag{1.7}$$

Repeated differentiation with respect to $\mathbf{b}$ then leads to the identity:

$$\langle x_{k_1}x_{k_2}\dots x_{k_\ell}\rangle = (2\pi)^{-n/2}(\det\mathbf{A})^{1/2}\left[\frac{\partial}{\partial b_{k_1}}\frac{\partial}{\partial b_{k_2}}\cdots\frac{\partial}{\partial b_{k_\ell}}I(\mathbf{A},\mathbf{b})\right]\Bigg|_{\mathbf{b}=0},$$

or replacing the integral $I(A,\mathbf{b})$ by its explicit form (1.5):

$$\langle x_{k_1}\dots x_{k_\ell}\rangle = \left\{\frac{\partial}{\partial b_{k_1}}\cdots\frac{\partial}{\partial b_{k_\ell}}\exp\left[\sum_{i,j=1}^{n}\tfrac{1}{2}b_i\left(A^{-1}\right)_{ij}b_j\right]\right\}\Bigg|_{\mathbf{b}=0}. \tag{1.8}$$

Wick's theorem. This identity leads to Wick's theorem. In the r.h.s. of equation (1.8) each time a differential operator acts on the exponential it generates a factor b. Another differential operator has to act on this factor, otherwise the corresponding contribution vanishes when we set $\mathbf{b} = 0$. We conclude that the average of the product $x_{k_1}\dots x_{k_\ell}$ with the gaussian weight $\exp\left(-\tfrac{1}{2}x_iA_{ij}x_j\right)$ is obtained in the following way: one considers all possible pairings of the indices $k_1,\dots,k_\ell$ (ℓ must thus be even). To each pair k_pk_q one associates the matrix element $\left(A^{-1}\right)_{k_pk_q}$ of the matrix $\mathbf{A}^{-1}$. Then:

$$\langle x_{k_1}\dots x_{k_\ell}\rangle = \sum_{\substack{\text{all possible pairings}\\ \text{of } (k_1\dots k_\ell)}} A^{-1}_{k_{p_1}k_{p_2}}\dots A^{-1}_{k_{p_{\ell-1}}k_{p_\ell}}. \tag{1.9}$$

Equation (1.9) which expresses Wick's theorem is, in the form adapted to Quantum Mechanics or Field Theory, the basis of perturbative calculations.

Remark. The gaussian integral has another remarkable property: if we integrate the exponential of a quadratic form over a subset of variables, the result is still the exponential of a quadratic form. This structural stability is related to some of the properties of the harmonic oscillator which will be discussed in Chapter 2.

1.2 Perturbation Theory

We now want to calculate the integral:

$$I = \int\prod_{i=1}^{n}\mathrm{d}x_i\,\exp\left(-\sum_{i,j=1}^{n}\tfrac{1}{2}x_iA_{ij}x_j - \lambda V(x)\right), \tag{1.10}$$

in which $V(x)$ is a polynomial in the variables x_i and λ is a parameter. We can expand the integrand in a formal power series in λ:

$$I = \int\prod_{i=1}^{n}\mathrm{d}x_i\,\exp\left(-\sum_{i,j=1}^{n}\tfrac{1}{2}x_iA_{ij}x_j\right)\left[\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}V^k(x)\right]. \tag{1.11}$$

Using equations (1.6,1.8) we can formally rewrite (1.10):

$$I = \left\{\exp\left[-\lambda V\left(\frac{\partial}{\partial b}\right)\right]\exp\left[\sum_{i,j=1}^{n}\tfrac{1}{2}b_i\left(A^{-1}\right)_{ij}b_j\right]\right\}\Bigg|_{\mathbf{b}=0}. \tag{1.12}$$

We can also directly calculate each term in the expansion using Wick's theorem (1.9).

Steepest-descent. In the case of contour integrals in the complex domain, one sometimes uses a method, steepest-descent, which reduces their evaluation to gaussian integrals. Let us consider the integral:

$$I = \int\prod_{i=1}^{n}\mathrm{d}x_i\,\exp\left[-\frac{1}{\lambda}S(x_1,\dots,x_n)\right]. \tag{1.13}$$

In the limit $\lambda\to 0$, the integral is dominated by saddle points $\{x_i^c\}$:

$$\frac{\partial S}{\partial x_i}\left(x_1^c,x_2^c,\dots,x_n^c\right) = 0. \tag{1.14}$$

To calculate the contribution of the leading saddle point $\mathbf{x}^c$, we change variables, setting:

$$\mathbf{x} = \mathbf{x}^c + \mathbf{y}\sqrt{\lambda}. \tag{1.15}$$

We then expand $S(\mathbf{x})$ in powers of λ (and thus y):

$$\frac{1}{\lambda}S(x_1,\dots,x_n) = \frac{1}{\lambda}S(\mathbf{x}^c) + \frac{1}{2!}\frac{\partial^2 S}{\partial x_i x_j}(\mathbf{x}^c)\,y_iy_j + \sum_{k=3}^{\infty}\frac{\lambda^{k/2-1}}{k!}\frac{\partial^k S}{\partial x_{i_1}\dots\partial x_{i_k}}(\mathbf{x}^c)\,y_{i_1}\dots y_{i_k}. \tag{1.16}$$

The change of variables is such that the term quadratic in $\mathbf{y}$ is independent of λ. The integral becomes:

$$I = \mathrm{e}^{-S(\mathbf{x}^c)/\lambda}\int\prod_{i=1}^{n}\mathrm{d}x_i\,\exp\left[-\frac{1}{2!}\frac{\partial^2 S}{\partial x_i\partial x_j}(\mathbf{x}^c)\,y_iy_j - R(\mathbf{y})\right] \tag{1.17}$$

$$R(\mathbf{y}) = \sum_{k=3}^{\infty}\frac{\lambda^{k/2-1}}{k!}\frac{\partial^k S}{\partial x_{i_1}\dots\partial x_{i_k}}(\mathbf{x}^c)\,y_{i_1}\dots y_{i_k}. \tag{1.18}$$

We then expand the integrand in powers of $\sqrt{\lambda}$: At each order we have to calculate the average of a polynomial with a gaussian weight.

1.3 Complex Structures

We shall often meet complex structures: we have $2n$ integration variables $\{x_i\}$ and $\{y_i\}$, $i = 1, \ldots, n$, and the integrand is invariant under a simultaneous identical rotation in all (x_i, y_i) planes. It is then natural to introduce formal complex variables z_i and $\bar{z}_i$ which, for normalization purposes, we define by:

$$z_i = (x_i + iy_i)/\sqrt{2}\,, \qquad \bar{z}_i = (x_i - iy_i)/\sqrt{2}\,. \tag{1.19}$$

Note however that z_i and $\bar{z}_i$ are *independent integration variables* and only formally complex conjugates since x_i and y_i could themselves be complex.

The generic gaussian integral now becomes:

$$I(\mathbf{A};\mathbf{b},\bar{\mathbf{b}}) = \int \left(\prod_{i=1}^{n} \mathrm{d}z_i \mathrm{d}\bar{z}_i\right) \exp\left[-\sum_{i,j=1}^{n} \bar{z}_i A_{ij} z_j + \sum_{i=1}^{n} \left(\bar{b}_i z_i + b_i \bar{z}_i\right)\right], \tag{1.20}$$

in which $\mathbf{A}$ is a complex matrix with non-vanishing determinant.

As before, to calculate this integral we first eliminate the terms linear in z_i and $\bar{z}_i$ by a shift of variables, setting:

$$z_i = v_i + \left(A^{-1}\right)_{ij} b_j\,, \qquad \bar{z}_i = \bar{v}_i + \bar{b}_j \left(A^{-1}\right)_{ji}\,. \tag{1.21}$$

The resulting gaussian integral can be calculated either by returning to the "real" variables (1.19) or by a change of variables like $A_{ij} v_j = v_i'$. We obtain

$$I(\mathbf{A};\mathbf{b},\bar{\mathbf{b}}) = (2i\pi)^n (\det \mathbf{A})^{-1} \exp\left[\sum_{i,j=1}^{n} \bar{b}_i \left(A^{-1}\right)_{ij} b_j\right]. \tag{1.22}$$

By systematically differentiating with respect to b_i and $\bar{b}_j$, one establishes Wick's theorem for averages with the gaussian weight $\exp(-\bar{z}_i A_{ij} z_j)$. Only monomials with equal number of factors z and $\bar{z}$ have a non vanishing average:

$$\left\langle \bar{z}_{i_1} z_{j_1} \ldots \bar{z}_{i_n} z_{j_n} \right\rangle = \sum_{\substack{\text{all permutations} \\ P \text{ of } \{j_1,\ldots,j_n\}}} A^{-1}_{j_{P_1} i_1} A^{-1}_{j_{P_2} i_2} \ldots A^{-1}_{j_{P_n} i_n}\,. \tag{1.23}$$

1.4 Integral Representation of Constraints

We shall often use a simple identity about Dirac δ-functions. By definition:

$$\int \prod_{i=1}^{n} \mathrm{d}y_i\, \delta(y_i) = 1\,. \tag{1.24}$$

If we change variables:

$$y_i = f_i(\mathbf{x}) \tag{1.25}$$

and assume that equation (1.25) defines a unique set of functions $x_i(\mathbf{y})$ for $|\mathbf{y}|$ small enough, then we obtain the identity:

$$\int \left\{\prod_{i=1}^{n} \mathrm{d}x_i\, \delta\left[f_i(\mathbf{x})\right]\right\} J(\mathbf{x}) = 1\,, \tag{1.26}$$

in which $J(\mathbf{x})$ is the jacobian of the change of variables (1.25):

$$J(\mathbf{x}) = \left|\det \frac{\partial f_i}{\partial x_j}\right|. \tag{1.27}$$

Identity (1.26) has a straightforward generalization: assume that we want to calculate a function $\sigma(\mathbf{x})$ for $\mathbf{x}$ solution of the equation $\mathbf{f}(\mathbf{x}) = 0$, i.e. for $\mathbf{x} = \mathbf{x}(\mathbf{y} = 0)$, without solving the equation explicitly. We can then use the identity:

$$\sigma(\mathbf{x})\big|_{\mathbf{f}(\mathbf{x})=0} = \int \left\{\prod_{i=1}^{n} \mathrm{d}x_i\, \delta\left[f_i(\mathbf{x})\right]\right\} J(\mathbf{x}) \sigma(\mathbf{x})\,. \tag{1.28}$$

This identity, as well as the identities about gaussian integrals, has the interesting property that they can be easily generalized to an infinite number of variables.

1.5 Algebraic Functional Techniques

1.5.1 Functional differentiation

In the discussion of algebraic properties of correlation functions the concept of generating functionals will be very useful. Let $f(x)$ be a function of a variable x, we shall consider objects of the form:

$$F(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n\, F^{(n)}(x_1, \ldots, x_n) f(x_1) \ldots f(x_n)\,, \tag{1.29}$$

in which $F^{(n)}(x_1, \ldots, x_n)$ is a symmetric function of its arguments. We shall also need the concept of functional derivative $\delta/\delta f(x)$. It is defined by the properties that it satisfies the usual algebraic rules of any differential operator:

$$\begin{aligned} \frac{\delta}{\delta f(x)}\left[F_1(f) + F_2(f)\right] &= \frac{\delta}{\delta f(x)} F_1(f) + \frac{\delta}{\delta f(x)} F_2(f), \\ \frac{\delta}{\delta f(x)}\left[F_1(f) F_2(f)\right] &= F_1(f) \frac{\delta}{\delta f(x)} F_2(f) + F_2(f) \frac{\delta}{\delta f(x)} F_1(f), \end{aligned} \tag{1.30}$$

and in addition:

$$\frac{\delta}{\delta f(y)} f(x) = \delta(x - y). \tag{1.31}$$

This implies for example:

$$\frac{\delta}{\delta f(y)} F(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n\, F^{(n+1)}(y, x_1, \ldots, x_n) f(x_1) \ldots f(x_n)\,. \tag{1.32}$$

1.5.2 Determinants of operators

Often we shall have to calculate determinants of operators represented by some kernel $M(x,y)$ which, after some transformations, can be cast into the form $\delta(x-y)+K(x,y)$. Provided the traces of all powers of K exist, the following identity, valid for any matrix M,

$$\ln\det \mathrm{M} \equiv \mathrm{tr}\ln \mathrm{M}\,, \tag{1.33}$$

expanded in powers of the kernel K:

$$\ln\det[1+\mathrm{K}] = \int \mathrm{d}x\, K(x,x) - \tfrac{1}{2}\int \mathrm{d}x_1 \mathrm{d}x_2\, K(x_1,x_2)K(x_2,x_1) + \cdots + \frac{(-1)^{n+1}}{n}\int \mathrm{d}x_1\cdots\mathrm{d}x_n\, K(x_1,x_2)\,K(x_2,x_3)\cdots K(x_n,x_1) + \cdots\,, \tag{1.34}$$

will often be useful.

1.6 Grassmann Algebras. Differential Forms

We shall also deal with theories containing fermions. Since fermion field correlation functions (or Green's functions) are antisymmetric with respect to the exchange of two arguments, the construction of generating functionals requires the introduction of anti-commuting classical functions, and thus Grassmann variables.

Grassmann algebra. We only consider Grassmann algebras over $\mathbb{R}$ or $\mathbb{C}$ (real or complex). A Grassmann algebra $\mathfrak{A}$ is an algebra constructed from a set of generators θ_i and their anticommuting products:

$$\theta_i\theta_j + \theta_j\theta_i = 0 \qquad \forall i,j\,. \tag{1.35}$$

Note that as a consequence:

(i) all elements in a Grassmann algebra are first degree polynomials in each generator;

(ii) if the number n of generators is finite, the algebra forms a finite dimensional vector space on $\mathbb{R}$ or $\mathbb{C}$ of dimension 2^n.

$\mathfrak{A}$ is also a graded algebra in the sense that to any monomial $\theta_{i_1}\theta_{i_2}\ldots\theta_{i_p}$ we can associate an integer p counting the number of generators in the product.

Finally let us note that elements of $\mathfrak{A}$ are invertible if and only if their expansion as a sum of products of generators contains a term of degree zero which is invertible. For example the element $1+\theta$ is invertible, and has $1-\theta$ as inverse; however θ is not invertible.

Grassmannian parity. On the algebra $\mathfrak{A}$ we can implement a simple automorphism which is a reflection P defined by:

$$\mathrm{P}(\theta_i) = -\theta_i\,. \tag{1.36}$$

Then on a monomial of degree p, P acts like:

$$\mathrm{P}(\theta_{i_1}\ldots\theta_{i_p}) = (-1)^p\theta_{i_1}\ldots\theta_{i_p}\,. \tag{1.37}$$

The reflection P divides the algebra $\mathfrak{A}$ in two eigenspaces $\mathfrak{A}^{\pm}$ containing the even or odd elements

$$\mathrm{P}(\mathfrak{A}^{\pm}) = \pm\mathfrak{A}^{\pm}. \tag{1.38}$$

In particular $\mathfrak{A}^+$ is a subalgebra, the subalgebra of commuting elements.

Differential forms. An application of Grassmann algebras is the representation of differential forms. The language of differential forms will not be used often in this work. However it is interesting to here recall one concept, the exterior derivative of forms, whose generalization will appear in the context of BRS symmetry (see Chapter 16). Let us consider totally antisymmetric tensors $\Omega_{\mu_1,\ldots,\mu_l}(x)$, functions of n commuting variables x^μ. Associating n Grassmann variables θ^μ with x^μ, we can write the corresponding l-form Ω:

$$\Omega = \Omega_{\mu_1,\ldots,\mu_l}(x)\theta^{\mu_1}\ldots\theta^{\mu_l}\,, \tag{1.39}$$

where $l \le n$ otherwise the form vanishes.

One can define a differential operator d acting on forms:

$$\mathrm{d} \equiv \theta^\mu \frac{\partial}{\partial x^\mu}\,. \tag{1.40}$$

We note that if Ω is a l-form, $\mathrm{d}\Omega$ is a $l+1$-form (see Chapter 22 for details). One immediately verifies that d is *nilpotent*:

$$\mathrm{d}^2 = \theta^\mu \frac{\partial}{\partial x^\mu}\theta^\nu \frac{\partial}{\partial x^\nu} = 0\,, \tag{1.41}$$

because the product $\theta^\mu\theta^\nu$ is antisymmetric in $\mu \leftrightarrow \nu$.

We also recall that a form Ω which satisfies $\mathrm{d}\Omega = 0$ is called *closed* and a form Ω which can be written $\Omega = \mathrm{d}\Omega'$ is called *exact*. The property (1.41) implies that any exact form is closed.

Note that it is customary to write in the case of forms the generators of the algebra $\mathrm{d}x^\mu$ instead of θ^μ and to then use the $\wedge$ notation for the product to show that it is antisymmetric.

1.7 Differentiation in Grassmann Algebras

It is then useful to define differentiation in Grassmann algebras. A naive definition would be inconsistent due to the non-commutative character of the algebra. The problem can be solved in the following way: Considered as functions of a generator θ_i, all elements A of $\mathfrak{A}$ can be written

$$A = A_1 + \theta_i A_2\,,$$

after some commutations, where A_1 and A_2 do not depend on θ_i. Then by definition

$$\frac{\partial A}{\partial \theta_i} = A_2\,. \tag{1.42}$$

Note that the differential operator $\partial/\partial\theta_i$ is *nilpotent*: $(\partial/\partial\theta_i)^2 = 0$, like the form differentiation (see equation (1.41)).

Remark. The equation (1.42) defines a left-differentiation in the sense that the action of $\partial/\partial\theta_i$ consists in bringing θ_i on the left in a monomial and suppressing it. Similarly a right-differentiation could have defined by commuting θ_i to the right.

Chain rule. It is easy to verify that chain rule applies to Grassmann differentiation. If $\sigma(\theta)$ belongs to $\mathfrak{A}^-$ and $x(\theta)$ belongs to $\mathfrak{A}^+$ we can write:

$$\frac{\partial}{\partial\theta} f(\sigma,x) = \frac{\partial\sigma}{\partial\theta}\frac{\partial f}{\partial\sigma} + \frac{\partial x}{\partial\theta}\frac{\partial f}{\partial x}\,. \tag{1.43}$$