Homework 1 PY501 (Fall 2012)

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Reading: Brown and Churchill Chapters 1-4. Any text that treats complex analysis is suitable for additional reading.

Problems: Due Wednesday Sept 19

1. (a) Use an argument analogous to the one in lecture to derive the Cauchy-Riemann equations in polar coordinates.
(b) Let $f(z) = u(r,\theta) + iv(r,\theta)$ be analytic in a domain D. Show that $u(r,\theta)$ satisfies Laplace's equation in polar coordinates using (a).
(c) Let $u(r,\theta) = \ln r$. Show this satisfies Laplace's equation. Also, find its harmonic conjugate $v(r,\theta)$ using the Cauchy-Riemann equations.

2. $f(z)$ is the branch

$$z^{-1+i} = \exp\left[(-1+i)\log z\right] \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

and the contour C is the unit circle $z = e^{i\theta}$, $(0 \le \theta \le 2\pi)$.
Find

$$\int_C f(z)dz$$

3. Use the following method to derive the integration formula

$$\int_0^\infty e^{-x^2} \cos 2bx \, dx = \frac{\sqrt{\pi}}{2} e^{-b^2}$$

(a) Show that the sum of the integrals of e^{-z^2} along the lower and upped horizontal legs of the rectangular path in the figure can be written

$$2\int_0^a e^{-x^2} dx - 2e^{b^2} \int_0^a e^{-x^2} \cos 2bx dx$$

and that the sum of the integrals along the vertical legs on the left and right can be written

$$ie^{-a^2} \int_0^b e^{y^2} e^{-i2ay} dy - ie^{-a^2} \int_0^b e^{y^2} e^{i2ay} dy.$$

(b) Use the Cauchy-Gorsat theorem, what we learned about Gaussian integrals, and the observation that

$$\left|\int_0^b e^{y^2} \sin 2ay \, dy\right| \le \int_0^b e^{y^2} dy,$$

to obtain the desired formula by letting a go to infinity, $\alpha \to \infty$.

4. Let C be the unit circle $z = e^{i\theta}, (-\pi \leq \theta \leq \pi)$. First show that for any real constant

$$\int_C dz \frac{e^{az}}{z} = 2\pi i.$$

Rewrite the integral in terms of θ to derive the integration formula

$$\int_0^\pi d\theta \, e^{a\cos\theta} \cos(a \sin\theta) d\theta = \pi$$

5 . Show that

$$\int_0^\infty \frac{dx}{1 + x^2 + x^4} = \frac{\pi\sqrt{3}}{6}; \qquad \int_{-\infty}^\infty \frac{dx}{(5 + 4x + x^2)^2} = \frac{\pi}{2}.$$