Measuring the Fermi Surface

1. De Haas-van Alphen Effect

A powerful technique for measuring the Fermi Surface is based on the de Haas-van Alphen effect discovered by de Hass and van Alphen in 1930. In this effect, the magnetization, normalized by the applied field, of a sample of bismuth was found to oscillate with H^{-1}. Later experiments have reproduced the oscillatory effect in the measurements of magnetic susceptibility, $\chi = dM/dB$.

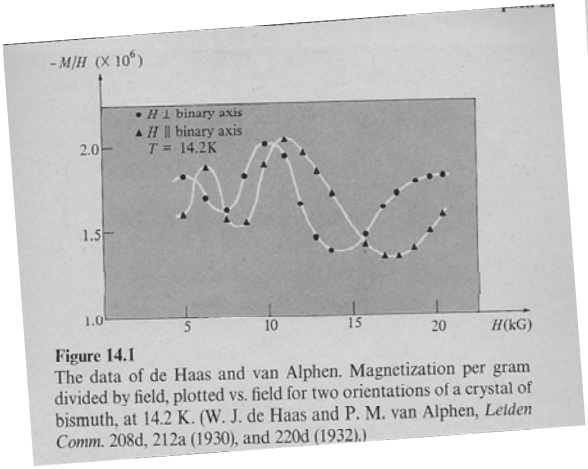


Figure 14.1
The data of de Haas and van Alphen. Magnetization per gram divided by field, plotted vs. field for two orientations of a crystal of bismuth, at 14.2 K. (W. J. de Haas and P. M. van Alphen, *Leiden Comm.* 208d, 212a (1930), and 220d (1932).)

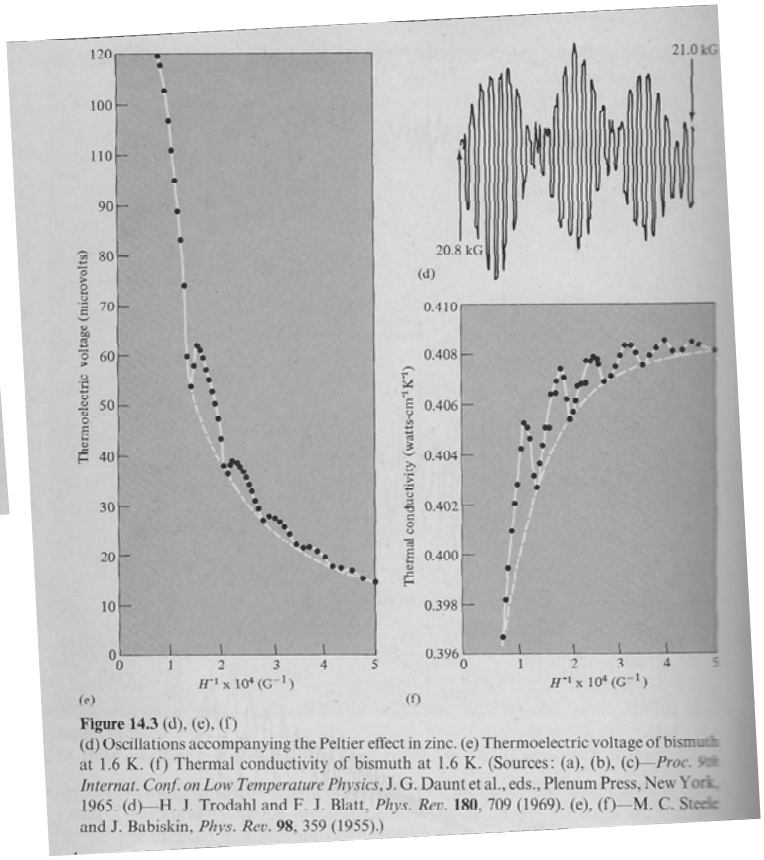


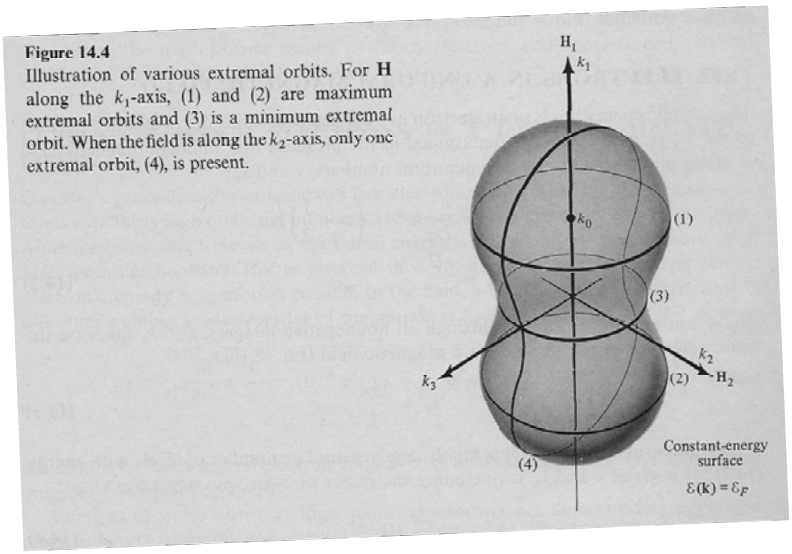
Figure 14.3 (d), (e), (f)
(d) Oscillations accompanying the Peltier effect in zinc. (e) Thermoelectric voltage of bismuth at 1.6 K. (f) Thermal conductivity of bismuth at 1.6 K. (Sources: (a), (b), (c)—*Proc. 9th Internat. Conf. on Low Temperature Physics*, J. G. Daunt et al., eds., Plenum Press, New York, 1965. (d)—H. J. Trodahl and F. J. Blatt, *Phys. Rev.* **180**, 709 (1969). (e), (f)—M. C. Steele and J. Babiskin, *Phys. Rev.* **98**, 359 (1955).)

Similar oscillatory behavior has been observed not only in magnetic susceptibility, but also in the conductivity for which the effect is referred to as the Shubnikov-de Haas effect, the magnetostriction (dependence of sample size on magnetic field) and when measured with sufficient care, in almost all other quantities.

In 1952, Onsager pointed out that the change in 1/B through a single period of oscillation is determined by:

$$\Delta\left(\frac{1}{B}\right) = \frac{2\pi e}{\hbar}\frac{1}{A_e} \quad . \quad (1)$$

where A_e is any extremal cross-sectional area of the Fermi surface in a plane normal to the B field.



Since altering the magnetic field direction brings different extremal areas into play, all extremal areas of the Fermi surface can be mapped out. In practice, rather than disentangling the geometrical information from the data, it is often easier to guess at what the surface is then later refine the guess by testing it against the data.

In the semiclassical model, the electron states are quantized in units of $\Delta^3 k = (2\pi)^3/L^3$. But (1) requires the degenerate states involved in the closed orbits for the electronic motion projected on a plane perpendicular to the field to be quantized differently. Observation of the dHvA effect thus shows that the semi-classical treatment is inadequate.

2. Free Electrons in a Uniform Magnetic Field

To find the energy levels of electrons in a magnetic field, one must in principle return to the Schrodinger equation for an electron in the periodic crystalline potential in the presence of the magnetic field. The full solution of this problem is a formidable task, which has been accomplished only in the simple case of free electrons (i.e., zero periodic potential) in a magnetic field. The results are presented below below.

The orbital energy levels of a free electron in a cubical box with sides of length L parallel to the x-, y- and z-axes are determined in the presence of a uniform magnetic field B along the z-direction by two quantum numbers, ν and k_z:

$$\varepsilon_\nu(k_z) = \frac{\hbar^2}{2m} k_z^{\,2} + \left(\nu + \frac{1}{2}\right)\hbar\omega_c, \tag{2}$$

where $\omega_c = \dfrac{eB}{m}$, $\nu = 0, 1, 2, \ldots,$ and

$$k_z = \frac{2\pi n_z}{L}, \text{ with } n_z = 0, \pm 1, \pm 2, \ldots \tag{3}$$

Equation (2) says that the energy of motion perpendicular to the field, which would be $\hbar^2(k_x^2 + k_y^2)/2m$ if no field were present, is quantized in steps of $\hbar\omega_c$.

The set of all levels with the same ν (and arbitrary k_z) is collectively called the ν^{th} Landau levels.

Each level is highly degenerate. The number of levels with energy given by eqn. (2) for a given ν and k_z (including the factor of 2 for spin degeneracy) is:

$$N_0 = (2e/h)BL^2 = BL^2/\Phi_0 \qquad (4)$$

where $$\Phi_0 \equiv h/2e = 2.067\text{e-}15 \text{ Wb} \qquad (5)$$

is a magnetic flux quantum. Equation (4) shows that the degeneracy of a Landau level is independent of ν and k_z. It only depends on the cross-sectional area of the specimen.

3. Levels of Bloch Electrons in a Uniform Magnetic Field

As discussed earlier, to solving the Schrodinger equation for this problem is a formidable task if the crystal potential is not zero. To treat the Bloch electrons, where the crystal potential is not zero, we adopt the approach used by A&M. As shown below, the Landau levels at the Fermi energy correspond to fairly high quantum numbers, ν. With this, one may use the correspondence principle in quantum mechanics, where $\Delta\varepsilon = \varepsilon_{\nu+1} - \varepsilon_\nu$ can be related to the semi-classical value $2\pi\hbar/T$, where T is the period of orbital motion derived earlier for Bloch electrons in the semi-classical treatment. Following this, one may discuss how the dHvA oscillations occur.

We first justify the use of the correspondence principle, which states that when the action of a system is much bigger than $\hbar$, it behaves classically. For electrons in a magnetic field, that can be translated to requiring that the quantum number ν of the Landau level is >> 1.

Landau level energy per magnetic field, $\hbar\omega_c/B = e\hbar/m_e = \hbar/m_e$ eV/T = 1.16e-4 eV/T.

As it is those states near the Fermi energy contribute to the transport properties, we consider the levels with energy $\approx \varepsilon_F$. Since ε_F is typically several eV, even in fields as high as 1 T, ν will be of order 1e4.

Energies of levels with very high quantum numbers can be accurately calculated with Bohr's correspondence principle: The difference in energy of two adjacent levels is Planck's constant times the frequency of classical motion at the energy of the levels. Since k_z is a constant of the semiclassical motion, we apply this condition to levels with a specific kz and quantum numbers ν and ν+1.

$$\varepsilon_{\nu+1}(k_z) - \varepsilon_\nu(k_z) = h/T(\varepsilon_\nu(k_z), k_z), \quad (6)$$

where $T(\varepsilon_\nu(k_z), k_z)$ is the period of semiclassical motion on the orbit specified by ε and kz. Note that $T = 2\pi/\omega_c$ only if the cyclotron effective mass is used for ω_c. However, the estimated value obtained by assuming $m^* = m_e$ usually gives within ±2 the correct order of magnitude.

$$T(\varepsilon_\nu(k_z), k_z) = (\hbar^2/eB)\partial A(\varepsilon,k_z)/\partial\varepsilon, \quad (7)$$

and $A(\varepsilon,k_z)$ is the k-space area enclosed by the orbit. Combine (6) and (7), we get:

$$(\varepsilon_{\nu+1}(k_z) - \varepsilon_\nu(k_z))\partial A(\varepsilon_\nu)/\partial\varepsilon = 2\pi eB/\hbar \quad (8)$$

Because we are interested in ε_ν on the order of ε_F, we expect that the energy difference between neighboring Landau levels will be of order $\hbar\omega_c$, which is ~1e4 times smaller than the energies of the levels themselves. It is thus a good approximation to take:

$$\partial A(\varepsilon_\nu)/\partial\varepsilon = [A(\varepsilon_{\nu+1}) - A(\varepsilon_\nu)]/[\varepsilon_{\nu+1} - \varepsilon_\nu] \quad (9).$$

Placing this in (8), we find:

$$\Delta A = A(\varepsilon_{\nu+1}) - A(\varepsilon_\nu) = 2\pi eB/\hbar \quad (10).$$

At B = 1 T, $\Delta A = 9.53e15\ m^{-2}$.

Another way of stating (10) is that, at large ν,

$$A(\varepsilon_\nu(k_z), k_z) = (\nu + \lambda)\Delta A, \quad (11).$$

where λ is a real number independent of ν.

4. Origin of the Oscillatory Phenomena

It turns out the quantization condition (11) causes the electronic density of levels, g(ε) to peak whenever ε is equal to the energy of an extremal orbit, as illustrated by the cartoon shown in Fig. 14.5. In panel (a) of this figure, a "Landau tube" is shown, containing the electrons orbits in the ν^{th} Landau level (so their areas are constant given by eqn. (11)) lying on constant energy surfaces $\varepsilon = \varepsilon_\nu(k_z)$. The contribution to g(ε) from the orbits on the Landau tube will be the no. of such levels with energies between ε and ε+dε. This in turn is proportional to the area of the portion of the tube contained between the constant-energy surfaces of energies ε and ε+dε. Fig. 14.5b shows this portion of tube when the energy ε of the orbits in the portion are not extremal (i.e., they do not intersect

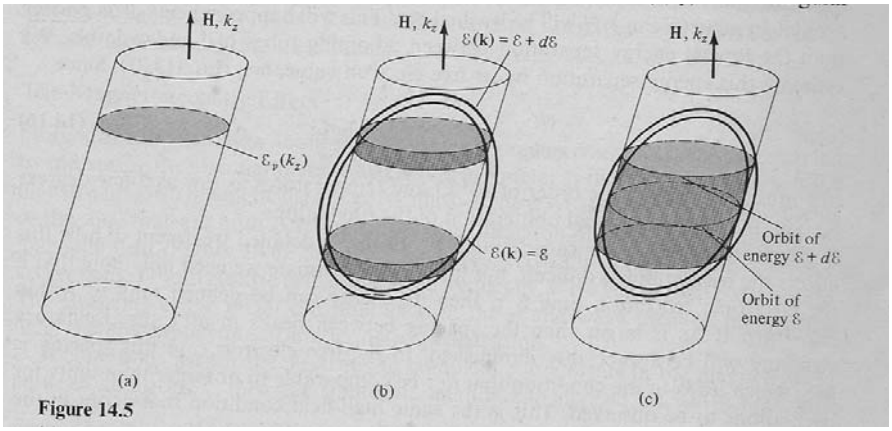


Figure 14.5

the extremal of the $\varepsilon_k = \varepsilon$ constant energy surface). Figure 14.5c shows the portion of the tube when there is an extremal orbit of energy ε on the tube. Evidently, the area of the portion of the tube is enormously enhanced in the latter case due to the very slow energy variation of levels along the tube near the extremal orbit.

Most electronic properties of metals depend on the density of levels at the Fermi energy, $g(\varepsilon_F)$. It follows from the above discussion that $g(\varepsilon_F)$ will peak whenever the value of the magnetic field causes an extremal orbit to overlap with the Fermi surface. Suppose these extremal orbits have quantum labels ν and k_z. The condition for $g(\varepsilon_F)$ to peak is that the area of these orbits, $A(\varepsilon_\nu(k_z), k_z)$ is equal to $A_e(\varepsilon_F)$, the extremal area of the Fermi surface. In other words, $(\nu + \lambda)\Delta A = A_e(\varepsilon_F)$. Substitute Eqn. (10) for ΔA in here, we find:

$$(\nu + \lambda)(2\pi eB_\nu/\hbar) = A_e(\varepsilon_F) \Rightarrow (\nu + \lambda)(2\pi e/(\hbar A_e(\varepsilon_F))) = 1/\, B_\nu \qquad (12)$$

Since $\Delta\nu = 1$, (12) give: $2\pi e/(\hbar A_e(\varepsilon_F)) = \Delta(1/\, B)$

$$\text{or, } \Delta(1/\, B) = (2\pi e/\hbar)\,(1/A_e(\varepsilon_F)). \qquad (13)$$

As the RHS of this equation is a constant, this explains the 1/B oscillatory phenomena.

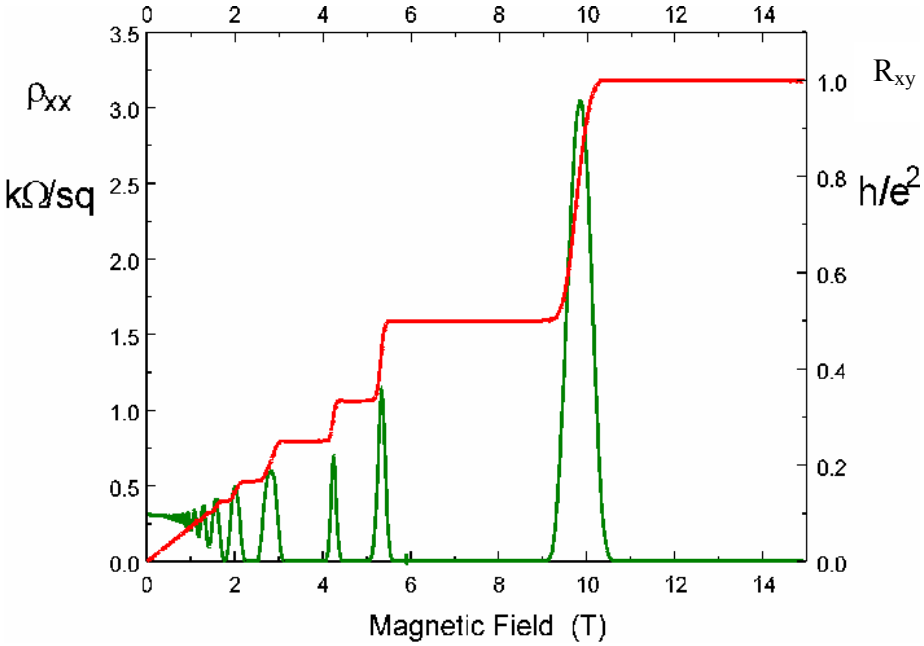
Note that for the oscillations to be seen, the temperature must be low enough for $k_BT < \hbar\omega_c$ to be valid. Validity of this condition can be assessed by the quantity:

$$e\hbar/mk_B = 1.34\ K/T.$$

This shows that for a 10 T field, the temperature can only be a few degrees K to avoid the Landau levels from being washed out by thermal agitations.

5. Quantum Hall Effect

Quantum Hall Effect is not a Fermi surface probe. But as in the dHvA effect it also arises from quantization of the electronic states into Landau levels. In 1985 Klaus von Klitzing won the Nobel Prize for the discovery of the Quantum Hall effect in a 2D electronic system. The major features of the phenomenon are depicted in the figure below.



http://www.warwick.ac.uk/~phsbm/qhe.htm

In this figure the plateau indices are, from right to left, 1, 2, 3, 4, 6, 8.... Odd integers correspond to the Fermi energy being in a spin gap and even integers to an orbital LL gap. As the spin splitting is small compared to the LL gaps, the odd integer plateaus are only seen at the highest magnetic fields (or lowest values of ν). The units of resistance, $h/e^2 \approx 25k\Omega$ is called 1 klitzing.

In Hall effect, the Hall coefficient is defined to be $R_H = E_y/Bj_x$, where $j_x = v_x nq$. We have shown that $R_H = 1/nq$. The Hall resistivity, ρ_{xy}, defined to be E_y/j_x is thus, B/nq.

For free electrons in 2D, the density of states is $g(\varepsilon) = m/\pi\hbar^2$, a constant. When a B field is applied, the available states clump into LL separated by the cyclotron energy, with regions of energy between the LLs where there are no allowed states. As the B field is increased, the LLs move relative to the Fermi surface as shown in the figure below.

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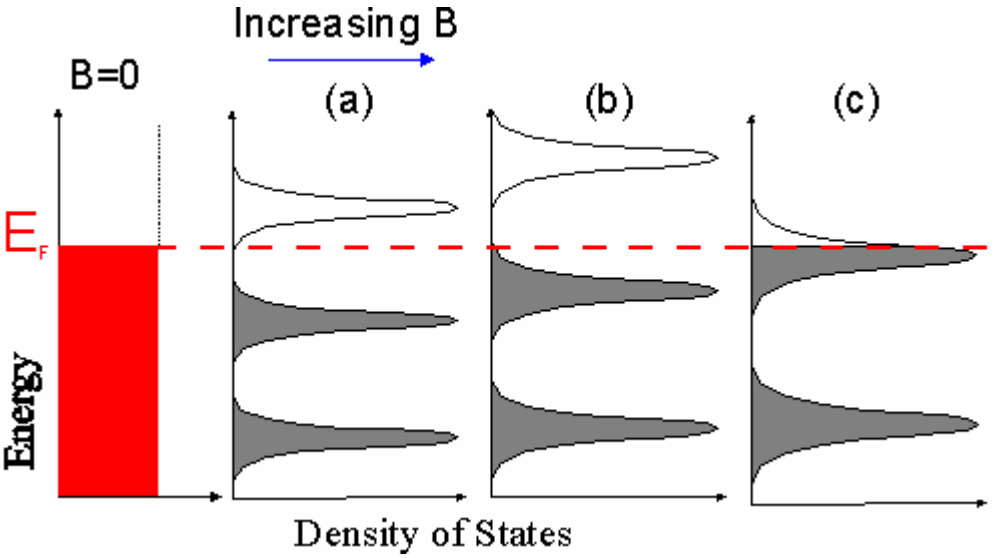

http://www.warwick.ac.uk/~phsbm/qhe.htm

Because of Zeeman splitting, each LL splits into a pair, one for spin up and one for spin down. The energy difference between the split levels is $2\mu_B B \approx e\hbar B/m$ and has the same energy scale as that of the LLs. It can be shown that each split LL has a degeneracy of eB/h per sample area, i.e., half of the value for the case without splitting.

Given the above background, we may now discuss the original of the observed quantization in R_{xy} (i.e., $R_{xy} = (h/(e^2 i) \sim 1/i$, where i is an integer). The number of current-carrying states per sample area in each LL per unit area of the sample is eB/h. So, if there are i LLs at energies below the Fermi energy, the density of electrons (i.e., number of electrons per unit volume) that can contribute to ρ_{xy} is (eB/h)(i)/d, where d is the thickness of the sample. So, $\rho_{xy} = B/ne = hd/(e^2 i)$. The Hall resistance, $R_{xy} = \rho_{xy}L/(Ld) = h/(e^2 i)$, where L is the length of the sample in the transverse direction.

Lastly, we examine the origin of zero resistivity, ρ_{xx}. That $\rho_{xx} = 0$ means that the resistivity tensor can be written as:

$$\rho = \begin{pmatrix} 0 & \rho_{xy} \\ \rho_{xy} & 0 \end{pmatrix}.$$

Correspondingly, the conductivity tensor is:

$$\sigma = \rho^{-1} = \frac{1}{-\rho_{xy}^{\,2}} \begin{pmatrix} 0 & -\rho_{xy} \\ -\rho_{xy} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \rho_{xy}^{\,-1} \\ \rho_{xy}^{\,-1} & 0 \end{pmatrix}.$$

One may thus perceive the peculiar behavior of $\rho_{xx} = 0$ found in QHE is a mathematical consequence of σ_{xx} being small and dominated by σ_{xy}.

Note that $\mathbf{E} = \boldsymbol{\rho}\mathbf{j} = \begin{pmatrix} 0 & \rho_{xy} \\ \rho_{xy} & 0 \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} \rho_{xy} j_y \\ \rho_{xy} j_x \end{pmatrix}$.

Appendix: Inverse matrix of a 2x2 matrix:

For $\mathrm{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse can be found using this formula:

$$\mathrm{A}^{-1} = \frac{1}{\det \mathrm{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$