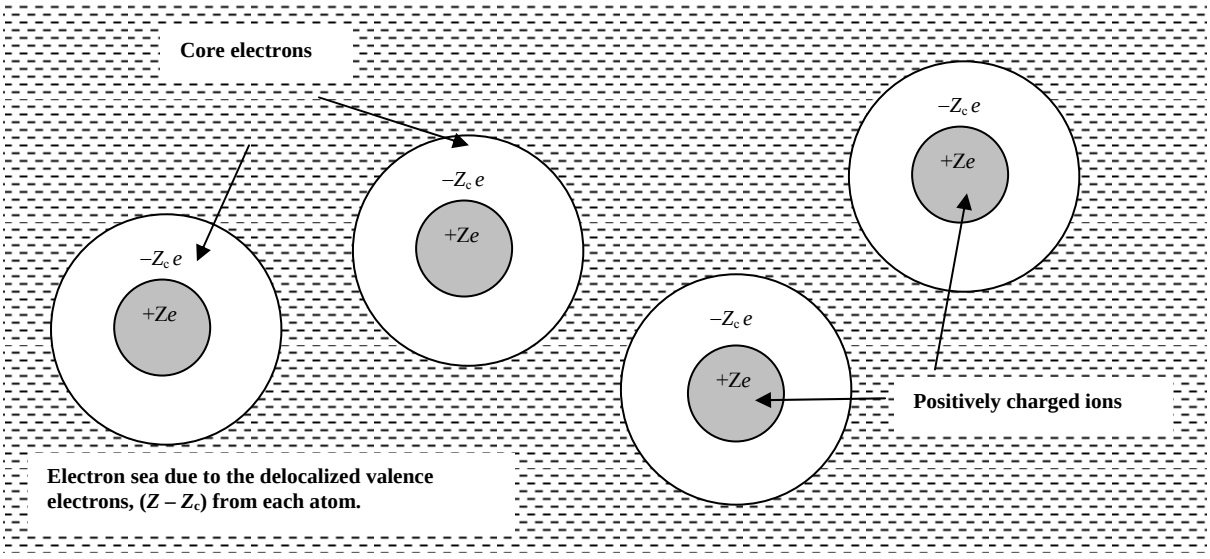
Drude Model

In 1897, J. J. Thomson discovered electrons.
In 1905, Einstein interpreted the photoelectric effect
In 1911

- Rutherford proved that atoms are composed of a point-like positively charged, massive nucleus surrounded by a sea of electrons.
- Drude constructed his theory of electrical and thermal conduction in metals by (1) considering the electrons to be a gas of negatively charged particles traversing in a medium of uniformly distributed positive ions, and (2) applying the kinetic theory of gas to the electron sea. Below is a schematic diagram of Drude's model of metals:

(In 1922, Bohr was awarded the Nobel Prize for his contribution to the understanding of the structure of atoms. In late 1925, the Schrödinger equation was formulated.)



There is one important parameter of the model:

(1) Electron number density, n

$$n = N/V = \underbrace{6.02 \times 10^{23}}_{\text{Avogadro's number}} \times (Z - Z_c)\rho_m/A, \qquad (1.1)$$

where N is the total number of electrons in the metal, V is the volume, ρ_m is the mass density, and A is the mass number. Note that only the valence electrons ($(Z - Z_c)$ per atom) contribute to n.

(2) Average electron separation, r_s.

$$V/N = 1/n = (4\pi/3)\, r_s^3$$
$$r_s = [3/(4\pi n)]^{1/3} \qquad (1.2)$$

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The value of n varies from $0.91\times10^{22}/\text{cm}^3$ for Cs to $24.7\times10^{22}/\text{cm}^3$ for Be among different metals. The values of r_s also vary between those of these two metals, from $5.62a_0$ to $1.87a_0$ (where a_0 is the Bohr radius = $4\pi\varepsilon_0\hbar^2/me^2 = 0.529\times10^{-8}$ cm), with the majority lying between $2a_0$ and $3a_0$.

You may notice the very high electron density found in metals, which is about 1000 times that of classical gases at normal temperature and pressure. In spite of this and in spite of the foreseeable strong electron-electron (e-e) and electron-ion (e-ion) electrostatic interactions, Drude boldly treated the dense electron gas by the kinetic theory of gas. Below are the basic assumptions he used:

1. Between collisions, the interaction of a given electron, both with other electrons and with the ions, is negligible. The former approximation is known as the *independent electron approximation* while the latter is the *free electron approximation*. Correspondingly, in the absence of externally applied electric or magnetic fields, each electron is taken to move in a straight line between collisions. In the presence of an applied field, however, each electron is taken to move according to the Newton's laws of motion.
2. Collisions are considered to be instantaneous events that alter the velocity of an electron. Drude attributed them to collisions with the presumably stationary ions (rather than collisions with other electrons as in ordinary gases). It turns out e-e scattering is indeed one of the least important of the several scattering mechanisms in a metal under normal conditions. However, the picture that e-ion scattering is the major scattering mechanism is also not accurate.
3. An electron is assumed to experience a collision with a probability per unit time of $1/\tau$. In other words, the probability of an electron undergoing a collision in an infinitesimal time interval, dt, is just dt/τ. The time τ is variously known as the relaxation time, collision time, or mean free time. In Drude's model, τ is assumed to be independent of the electron position and is independent of time.
4. Electrons are assumed to achieve thermal equilibrium with their surroundings only through collisions. In particular, immediately after each collision, an electron is taken to emerge with a velocity totally uncorrelated with its velocity before collision. Moreover, it is randomly directed and assumes a magnitude accountable by local thermal equilibrium at the location where the collision took place (i.e., $mv_{final}^2/2 = (3/2)k_BT_{local}$). Thus the hotter the region in which a collision occurs, the faster an electron will emerge on average from the collision.

In the following, we shall examine how Drude used these assumptions to build models predicting the electronic transport properties of metals, and how well the predictions describe the experiments or reality.

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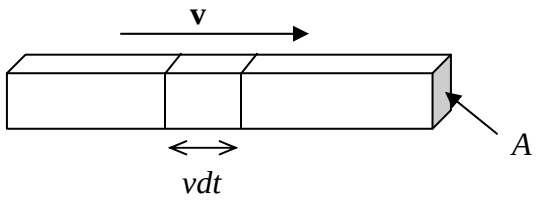
1.1 DC Electrical Conductivity of a Metal

According to Ohm's law:

$$\mathbf{E} = \rho\mathbf{j},$$

where **j** is the current density (= current per unit area). Note that it is a vector quantity. **E** is the electric field (= V/L where V is the applied voltage difference giving rise to the current, and L is the distance over which the voltage difference is applied.)

To relate **j** to the average velocity, **v**, of the electrons, consider a metal wire, with a cross-sectional area, *A*, and electron number density *n*, and suppose that the electrons in the wire are moving at a uniform velocity, **v**, as shown below:



After a given time lapse, *dt*, the electrons would have traversed a distance of *vdt* through the wire. This gives the number of electrons crossing an area *A* in time *dt* to be *nvA dt.* Since each electron carries a charge of – *e*, this crossing of electrons gives rise to a current density (i.e. charge per unit area per unit time):

$$\mathbf{j} = -en\mathbf{v}. \tag{1.3}$$

Next, we examine how **j** is related to **E** due to Drude's assumptions. Consider an arbitrary electron, and let *t* be the time elapsed since its last collision. Suppose its velocity right after the last collision is $\mathbf{v}_o$. After time *t*, this electron would have accumulated a velocity of $\mathbf{v}_o - e\mathbf{E}t/m$ based on Newton's second law. Since Drude assumed that the electrons emerge in random directions from a collision, there will be no contribution to the average velocity from $\mathbf{v}_o$. It follows that the current density must come entirely from the average of $-e\mathbf{E}t/m$. Denote the average elapsed time between collisions or the relaxation time by τ, we find from eqn. (1.3) that

$$\mathbf{j} = -en\mathbf{v}_{\mathbf{avg}} = (ne^2\tau/m)\mathbf{E}. \tag{1.4}$$

Using Ohm's law, and that electrical conductivity, $\sigma = 1/\rho$, we have:

$$\mathbf{j} = \sigma\mathbf{E};\ \ \sigma = ne^2\tau/m. \tag{1.5}$$

Rearranging this equation, we have $\tau = m/\rho ne^2$. Typical values of ρ for metals at room temperature are of the order of μOhm-cm. Therefore, it is more convenient to express τ in terms of ρ_μ in μOhm-cm instead of ρ (in Ohm-m) as follows:

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$$\rho = \rho_\mu \ \mu\Omega\text{cm} = \rho_\mu \times 10^{-6} \times 0.01\,\Omega\text{m} = \rho_\mu \times 10^{-8}\,\Omega\text{m}$$

$$\therefore \quad \tau = \frac{m}{\rho n e^2} = \frac{9.1\times10^{-31}\ \text{kg}}{(1.6\times10^{-16})^2\ \text{C}^2} \times \frac{1}{\rho_\mu \times 10^{-8}\,\Omega\text{m}} \times \frac{4}{3}\pi\left(\frac{r_S}{a_0}\right)^3 \times (0.5\times10^{-10}\,\text{m})^3$$

$$= 0.2\times10^{-14}\,\text{s} \times \frac{1}{\rho_\mu} \times \left(\frac{r_S}{a_0}\right)^3$$

This result suggests that the relaxation time is of the order of 10^{-15} to10^{-14} sec. To see whether this is a reasonable estimate, we use it to estimate the mean free path, l ($\cong v_0\tau$). Using the equipartition law, $(½)mv_0^2 = (3/2)k_BT$, where k_B is the Boltzmann constant and T is absolute temperature, we have

$$v_0 = \sqrt{\frac{3k_BT}{m}} = \sqrt{\frac{3\times1.38\times10^{-23}\ \text{JK}^{-1}\times300\ \text{K}}{9.1\times10^{-31}\ \text{kg}}}$$

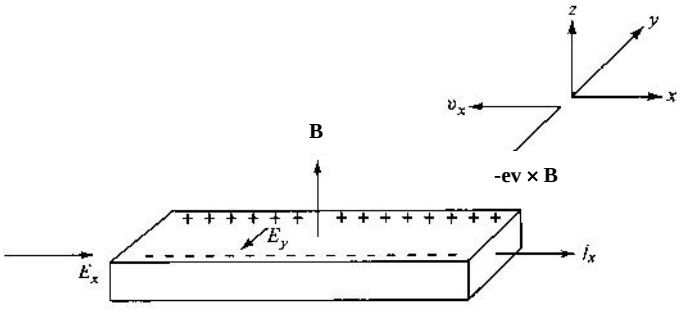
$$\approx 1 \times 10^5\ \text{ms}^{-1}$$

$$= 1 \times 10^7\ \text{cms}^{-1}$$

This gives $l \approx 1 - 10$ Å at $T = 300$K. Note that the additional contribution to v from E is $-eE\tau/m$ (~ 1.7 cm/s), thus is negligible. At lower temperatures, the Drude model predicts a smaller still value of l. In later chapters, you will see that the above estimate of v_0 is an order of magnitude less than the actual value at room temperatures. Furthermore, at very low temperatures, v_0 gets 10 times bigger than the room-temperature value and is independent of T. This raises the mean free path to >1000 Å, i.e., ~1000 times the inter-ionic spacing. This is strong evidence that the electrons do not simply bump off the ions. In the absence of a theory of the electron collisions and hence the relaxation time, τ, it becomes important to find predictions of the Drude model that are independent of the value of τ. There are several τ-independent quantities that are derivable by the Drude model and still of fundamental interest today. Below, we shall discuss three such quantities, namely the dc/ac electrical conductivities in the presence of a (spatially) uniform dc magnetic and a uniform dc/ac electric field and the ratio of the thermal to electrical conductivities (Wiedemann-Franz law).

1.2 Hall Effect and Magnetoresistance

Consider a metal block subject to an externally applied uniform electric field pointing in the +*x* direction (= E_x ***x***), and a uniform magnetic ***B*** field pointing in the +*z* direction as shown below (Hall's experiment, 1879):



Suppose that the applied electric field brings about an electronic current density of j_x in the +*x* direction. In the presence of the applied magnetic field ***B***, each electron constituting the current experiences a Lorentz force of

$$-e\boldsymbol{v} \times \boldsymbol{B}$$

(= $-evB(-\mathbf{x} \times \mathbf{z}) = -evB\,\boldsymbol{y}$) acting along the –y-direction. Since the two sides of the metal block along *y* are not connected, electrons deflected by the Lorentz force towards the –*y* side cannot escape, resulting in an accumulation of the electrons there. Because of electrical neutrality, there must be the same amount of positive charges accumulated on the +*y* side of the block, leading to an electric field, E_y, pointing in the –y-direction. In the steady state, the field E_y stops further deflection of the electrons by the Lorentz force. This happens when the force from E_y equals to the Lorentz force or $E_y = vB$.

There are two quantities of interest. One is the magnetoresistance, ρ(*B*):

$$\rho(B) = E_x/j_x. \tag{1.6}$$

The other quantity is the Hall coefficient, R_H:

$$R_H = E_y/(j_x B). \tag{1.7}$$

Notice that for electrons, $\boldsymbol{E}_y$ points in the –y-direction and R_H is negative. But if the moving charges were positive, which was unthinkable in Drude's time, the Lorentz force (now = +*e*vB ($\mathbf{x} \times \mathbf{z}$)) would still be pointing in the –y-direction. But because the moving charges are positive, the Lorentz force leads to accumulation of positive charges in the –y side, and negative charges in the +y side of the metal block. This causes the transverse field $\boldsymbol{E}_y$ to be directed in the +y-direction and the Hall coefficient R_H to be positive. It

turns out some metals (e.g. Be, Mg, In, Al) exhibit positive values of R_H at a large magnetic field of 1 Tesla.

To proceed, one observes that the average velocity, $\mathbf{v}(t)$ of the electrons at time t is related to its average momentum, $\mathbf{p}(t)$ by $\mathbf{v} = \mathbf{p}(t)/m$. We are aspired to calculate the average momentum of the electrons at a later time $t + dt$. If an electron does not encounter a collision before this time, it would be accelerated by the total force, $\mathbf{f}(t)$, arising from the applied E and B fields. According to Drude's assumptions, electrons as such amount to a fraction $(1 - dt/\tau)$ of the total. By using this and Newton's second law, we have

$$\begin{aligned}\mathbf{p}(t + dt) &= (1 - dt/\tau)[\mathbf{p}(t) + \mathbf{f}(t)\, dt + \mathrm{O}(dt)^2]. \\ &= \mathbf{p}(t) - (dt/\tau)\mathbf{p}(t) + \mathbf{f}(t)\, dt + \mathrm{O}(dt)^2. \end{aligned} \tag{1.8}$$

Note that we have neglected the contribution from the rest of the electrons that have undergone a collision within dt. It is because the directions of motion of these electrons would be randomized after the collision and so the average momentum (a vector) of these electrons would be zero. The correction term $\mathrm{O}(dt)^2$ comes from the fact that the force is actually varying from $\mathbf{f}(t)$ at time t to $\mathbf{f}(t + dt)$ at time $(t + dt)$ with the amount of variation being of order dt. Equation 1.8 can be rearranged to give:

$$\mathbf{p}(t + dt) - \mathbf{p}(t) = -(dt/\tau)\mathbf{p}(t) + \mathbf{f}(t)\, dt + \mathrm{O}(dt)^2. \tag{1.9}$$

Taking the limit $dt \to 0$, we have:

$$d\mathbf{p}(t)/dt = -\mathbf{p}(t)/\tau + \mathbf{f}(t). \tag{1.10}$$

This simply states that the effect of collisions is to introduce a frictional damping term to the equation of motion of the electrons. We may now use this equation to calculate the magnetoresistance and Hall coefficient. Given Hall's conditions, eqn. (1.10) can be rewritten as:

$$\frac{d\vec{p}}{dt} = -e\left(\vec{E} + \frac{\vec{p}}{m} \times \vec{B}\right) - \frac{\vec{p}}{\tau}. \tag{1.11}$$

In steady state, $\frac{dp}{dt} = 0$. Hence

$$0 = -eE_x - \frac{eB}{m} p_y - \frac{p_x}{\tau}. \tag{1.12}$$

$$0 = -eE_y + \frac{eB}{m} p_x - \frac{p_y}{\tau}. \tag{1.13}$$

Write $\omega_c = eB/m$, and multiply the above equations by $-ne\tau/m$, then use $\vec{j} = -ne\vec{p}/m$, one obtains:

$$\sigma_0 E_x = \omega_c \tau\, j_y + j_x, \tag{1.14}$$

$$\sigma_0 E_y = -\omega_c \tau \; j_x + j_y, \tag{1.15}$$

where σ_0 is the Drude model DC conductivity in the absence of a magnetic field. The Hall field is determined by the condition that there be no transverse current j_y. Setting $j_y = 0$ in eq. (1.15), we have:

$$\boxed{R_H = \frac{E_y}{j_x B} = -\frac{1}{ne}.} \tag{1.16}$$

Equation (1.16) predicts that R_H depends on no parameters of the metal, except the density of the charge carriers. Moreover, R_H should only be negative, as expected from Drude's presumption that the charge carriers are electrons. It turns out eqn. (1.16) provides good agreement with the alkali metal only. As pointed out above, some metals show positive R_H.

Appendix: Derivation of ω_c:

Consider an electron moving in a circle with radius r in the x-y plane, under a magnetic field B field pointing in +z.

$$\boldsymbol{F_B} = -e\boldsymbol{v} \times \boldsymbol{B} = -evB\boldsymbol{r}$$

The Lorentz force provides the centripetal force that sustains the circular motion:

$$evB = mv^2/r \Rightarrow eB = mv/r = m\omega \Rightarrow \omega = eB/m$$

We can similarly determine the magnetoresistance by taking $j_y = 0$ in eq. (1.14), which give:

$$\rho(B) = \frac{E_x}{j_x} = \frac{1}{\sigma_0}. \tag{1.17}$$

This result shows that the magnetoresistance is the same as the zero-field resistivity, which agrees to some degree with experiment.

1.3 AC Electrical conductivity of a metal

Consider an applied AC electric field of the form:

$$\vec{E}(t) = \vec{E}(\omega)\exp(-i\omega t). \tag{1.18}$$

The equation of motion for an electron under the influence of this electric field is:

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$$\frac{d\vec{p}}{dt} = -\frac{\vec{p}}{\tau} - e\vec{E}. \tag{1.19}$$

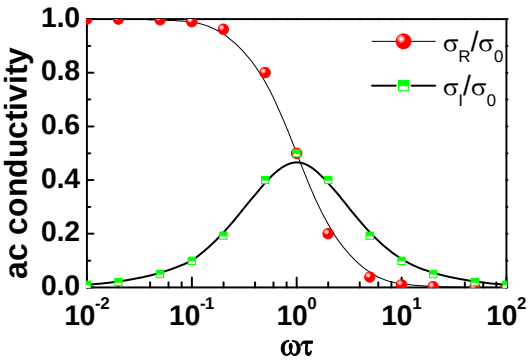
We assume a solution to Eq. (1.19) of the form:

$$\vec{p}(t) = \vec{p}(\omega)\exp(-i\omega t). \tag{1.20}$$

and substitute Eqs. (1.18) and (1.20) into Eq. (1.19). It leads to:

$$-i\omega\vec{p}(\omega) = -\frac{\vec{p}(\omega)}{\tau} - e\vec{E}(\omega)$$

$$\Rightarrow \vec{p}(\omega) = -\frac{e\vec{E}(\omega)}{1/\tau - i\omega}.$$



Further use $\mathbf{j}(\omega) = -ne\mathbf{p}(\omega)/m$, we have:

$$\vec{j}(\omega) = \frac{(ne^2/m)\vec{E}(\omega)}{1/\tau - i\omega}.$$

Writing this result in the form $\mathbf{j}(\omega) = \sigma(\omega)\mathbf{E}(\omega)$, we may find:

$$\boxed{\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}, \qquad \sigma_0 = \frac{ne^2\tau}{m}.} \tag{1.21}$$

Note that Eq. (1.21) correctly reduces to the Drude DC conductivity when frequency is equal to zero. An important application of this result is the propagation of EM waves in a metal. At first sight, it appears that Eq. (1.21) cannot be applicable since (1) the calculation does not account for the magnetic field, B(ω) of the EM wave, and (2) the derivation assumes the E(ω) field to be spatially uniform. For point (1), the effect of B(ω)is in fact negligible since $B(\omega)/E(\omega) \sim (\varepsilon(\omega)\mu(\omega))^{1/2} \sim 1/c$ (where c is the speed of light, ε(ω) the electric permittivity, and μ(ω) the magnetic permeability of the metal). Therefore, the ratio of the Lorentz force (due to B(ω)) to the electric force (due to E(ω)) is $\sim evB(\omega)/e\,E(\omega) \sim v/c$ (v is the average velocity of the electrons) ~ 0.01, hence is negligible. As for point (2), we recall that Eq. (1.19) assumes the **E** field to exercise its influence on the electron motion only over the distance between collisions. On average, this distance is of the order of the mean free path, l, which is 1 to 10 Å. Most of our discussions concern visible or UV lights, where the wavelength is 10^3 to 10^4 Å ($>> l$), so eqn. (1.21) is applicable.

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With the above result, we examine how an EM wave with wavelength much bigger than the electron mean free path propagates in a metal. Using Maxwell's equations, we have:

$$\begin{cases} \nabla\cdot\vec{E}=0, \quad \nabla\cdot\vec{B}=0, & (\nabla\cdot\vec{E}=0 \ \because \text{no net charge.}) \\ \nabla\times\vec{E}=-\partial\vec{B}/\partial t, & \\ \nabla\times\vec{B}=\mu_0\mu_r\vec{j}+(1/c^2)\partial\vec{E}/\partial t. & (\mu_r\approx 1 \text{ for most non-magnetic metals.}) \end{cases}$$

Substitute $\boldsymbol{B} = \boldsymbol{B}(\omega)\exp(-i\omega t)$, $\boldsymbol{E} = \boldsymbol{E}(\omega)\exp(-i\omega t)$ and $\boldsymbol{j} = \sigma(\omega)\boldsymbol{E}(\omega)\exp(-i\omega t)$ in the last two equations. We obtain:

$$\begin{cases} \nabla\times\vec{E}=i\omega\vec{B} \\ \nabla\times\vec{B}=(\mu_0\sigma(\omega)-i\omega/c^2)\vec{E}. \end{cases}$$

$$\therefore \ \underbrace{\nabla\times\nabla\times\vec{E}}=(i\omega\mu_0\sigma(\omega)+\omega^2/c^2)\vec{E}.$$

$$(=\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}\ =-\nabla^2\vec{\mathrm{E}} \ \because \ \nabla\cdot\vec{E}=0)$$

$$\Rightarrow -\nabla^2\vec{E}=\frac{\omega^2}{c^2}\left(1+\frac{i\sigma_0}{\varepsilon_0\omega}\cdot\frac{1}{1-i\omega\tau}\right)\vec{E}.$$

In the above, we have used $c = 1/(\varepsilon_0\mu_0)^{1/2}$, $\sigma(\omega) = \sigma_0/(1 - i\omega t)$ from Eq. (1.21). In the limit, $\omega\tau >> 1$, and use $\sigma_0 = ne^2\tau/m$, we have:

$$-\nabla^2\vec{E}=\frac{\omega^2}{c^2}\left(1-\frac{ne^2/m\varepsilon_0}{\omega^2}\right)\vec{E}. \tag{1.22}$$

Writing Eq. (1.22) in the form

$$-\nabla^2\vec{E}=\frac{\omega^2}{c^2}\varepsilon(\omega)\vec{E}, \tag{1.23}$$

and $\omega_p = ne^2/m\varepsilon_0$,

we have: $$\varepsilon(\omega)=1-\frac{\omega_p{}^2}{\omega^2}, \tag{1.24}$$

where ω_p is widely known as the plasma frequency.

Appendix: Derivation of eqn. (1.23) :

$\vec{j}=\dfrac{\partial\vec{p}}{\partial t}=\varepsilon_0(\varepsilon-1)\dfrac{\partial\vec{E}}{\partial t}$ (where ***p*** here denotes electric polarization, not momentum)

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$$\nabla \times B = \mu_0 \varepsilon_0 (\varepsilon - 1)\frac{\partial \vec{E}}{\partial t} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \frac{\varepsilon}{c^2}\frac{\partial \vec{E}}{\partial t} = \frac{-i\varepsilon\omega}{c^2}\vec{E}$$

$$-\nabla^2 \boldsymbol{E} = i\omega\nabla\times \boldsymbol{B} = (\varepsilon\omega^2/c^2)\boldsymbol{E}$$

If we now write $\boldsymbol{E} = \boldsymbol{E}(\omega,t)\exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$, and substitute it in Eq. (1.23), we obtain the usual dispersion relation:

$$k^2 = \varepsilon(\omega)\frac{\omega^2}{c^2}. \qquad (1.25)$$

Eq. (1.24) shows that for $\omega < \omega_p$, $\varepsilon(\omega)$ is negative, and by eq. (1.25) $k = ik_I$ is purely imaginary. That means $E \sim \exp(ikz) \sim \exp(-k_I z)$ and so decay exponentially in space (Here, without loss of generality we have assumed $\boldsymbol{k} = k\boldsymbol{z}$). But for $\omega > \omega_p$, $\varepsilon(\omega)$ is positive and $k = k_R$ is real, so $E \sim \exp(ikz) \sim \exp(ik_R z)$ can propagate inside the metal. Notice that this result is valid only if the above assumption, namely $\omega\tau >> 1$, holds in the neighborhood of ω_p. By using $1/n = 4\pi r_s^3/3$ and $r_s^3 = 2a_0$, we deduce that $\omega_p \sim 2.5 \times 10^{16}$ rad s^{-1}. ($k_p \sim \omega_p/c \sim 1\times 10^8$ $m^{-1} \Rightarrow \lambda = 2\pi/k = 6\times 10^{-8}$ m, in UV.) Recall that $\tau \sim 10^{-15}$ s. Hence, $\omega\tau \approx 25 >> 1$. Indeed Eq. (1.24) has been found to be obeyed by the alkali metals. But notice that in other metals, different contributions to the dielectric function competes with the "Drude term" shown in Eq. (1.24).

At $\omega = \omega_p$, $\varepsilon(\omega) = 0$ from Eq. (1.24). By Eq. (1.25), this causes $k = 0$ or $\lambda = 2\pi/k = \infty$. Therefore, the propagation mode with $\omega = \omega_p$ corresponds to a uniform oscillation of the electron sea in the metal against the positively charged background. This mode is known as the plasma oscillation.

Derive ω_p:

Consider the electron sea in a metal displaced uniformly by distance x in the +x direction. This causes accumulation of a surface charge density, $\sigma = -nex$ on the right side and $+nex$ on the left side of the metal.

$\nabla\cdot E = \rho/\varepsilon_0 \Rightarrow E = \sigma/\varepsilon_0$. (Notice that σ here denotes the surface charge, not electrical conductivity.)

Apply Newton's law, namely (mass)(acceleration) = force, we have

$m d^2\boldsymbol{x}/dt^2 = (-e)\boldsymbol{E} = (-e)(ne\boldsymbol{x}/\varepsilon_0)$ ($\boldsymbol{E}$ = (surface charge density)/ε_0 and is in the same/opposite direction as/to the surface normal if the surface charge is positive/negative.)

$d^2\boldsymbol{x}/dt^2 = -ne^2\boldsymbol{x}/(\varepsilon_0 m)$

$\omega_p = ne^2/(\varepsilon_0 m)$

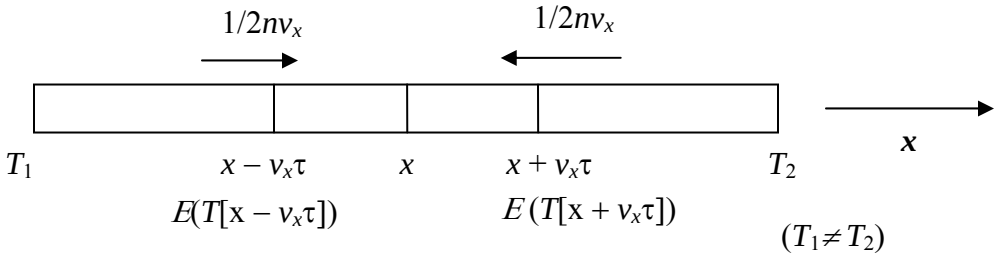
1.4 Thermal conductivity of a metal

When a metal bar is subject to a temperature gradient, ∇T, heat flows from the high temperature side to the low temperature side. If $|\nabla T|$ is small, the flow of thermal energy per unit area per unit time (= thermal current density, $\boldsymbol{j}^q$ (in $\mathrm{Wm^{-2}}$)) is related to ∇T by:

$$\boldsymbol{j}^q = -\kappa \nabla T, \tag{1.26}$$

where κ is the thermal conductivity (in $\mathrm{Wm^{-1}K^{-1}}$). Note that $\boldsymbol{j}^q$ is always in a direction opposite to ∇T because $\boldsymbol{j}^q$ is directed towards the low temperature side whereas ∇T increases towards the high temperature side. As a result, κ is always positive.

Without loss of generality, we choose ∇T and hence the heat current to be along x:



At any point x, electrons may migrate from either the $+x$ or $-x$ direction with number density per unit time of $+1/2nv_x$ and $-1/2nv_x$, respectively. Recall that the electrons, through collisions, acquire the *local* temperature at where they reside in. At point x, the most nearby collisions occur at $x - v_x\tau$ and $x - v_x\tau$ in the $-x$ and $+x$ direction, respectively. On arriving at x, the electrons bring in the thermal energy they just acquired from the last collision, i.e., $E(T[\mathrm{x} - v_x\tau])$ for those migrating from $-x$ and $E(T[\mathrm{x} + v_x\tau])$ for those from $+x$. The thermal current density is thus:

$$\boldsymbol{j}^q = nv_x/2\{\, E(T[\mathrm{x} - v_x\tau]) - E(T[\mathrm{x} + v_x\tau])\,\} \tag{1.27}$$

If $v_x\tau$ is very small, we may expand Eq. (1.27) about x:

$$\boldsymbol{j}^q = -nv_x^2\tau\,(d\mathrm{E}/dT)(dT/dx) \tag{1.28}$$

In 3-dimensions (3D), v_x in Eq. (1.28) would be the x-component of the average velocity of the electrons, which can have components along y and z. Since $\langle v_x^2\rangle = \langle v_y^2\rangle = \langle v_z^2\rangle = v^2/3$, and $n\,d\mathrm{E}/dT = c_v$, the electron specific heat capacity ($\mathrm{JK^{-1}m^{-3}}$), we have:

$$\boldsymbol{j}^q = (1/3)v^2\tau c_v\,(-\nabla T) \tag{1.29}$$

Hence, $$\kappa = (1/3)v^2\tau c_v = (1/3)lvc_v\,, \tag{1.30}$$

where v^2 is the mean square electron velocity, l is the mean free path.

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The most impressive success of the Drude model at the time was its ability to explain the empirical law of Wiedemann and Franz (W-F) (1853). The law states that the ratio, κ/σ for metals is directly proportional to T. In the following, we shall derive this law by applying the Drude model. Specifically, we apply eqs. (1.5) and (1.29) and find:

$$\frac{\kappa}{\sigma} = \frac{(1/3)c_v m v^2}{ne^2} \quad . \tag{1.30}$$

In the spirit of the Drude model, i.e., kinetic theory of electron gas, it is natural to assume $c_v = (3/2)nk_B$ and $(1/2)mv^2 = (3/2)k_BT$. This leads to the result:

$$\frac{\kappa}{\sigma T} = \frac{3}{2}\left(\frac{k_B}{e}\right)^2 = 1.11\times10^{-8}\ \text{W}\Omega/\text{K}^2. \tag{1.31}$$

The above ratio is called the Lorentaz number. Experimentally found values of the Lorentz number are typically ~2.5e-8 $\text{J}\Omega/\text{K}^2\text{s}$, which is in remarkable agreement with the Drude value. It turns out this agreement is a result of an incidental cancellation of two quantities in eq. (1.30), namely c_v and v^2, that are ~100 times off from their actual values. Notice that Drude had only considered the electronic contribution to the thermal conductivity. It was rationalized by the observation that the thermal conductivity of metals was usually higher than the insulators. In the next section, we briefly discuss the reason. However, it involves concepts that we won't cover until we discuss Lattice Dynamics.

Thermopower

In the above derivation, we have neglected the possibility that $\mathbf{v}(x - v\tau)$ and $\mathbf{v}(x + v\tau)$ may be different due to the temperature gradient. It turns out it is the correct to assume them to be the same. It is because the thermal conductivity measurement is conducted in an open circuit configuration, so any net flow of the electrons to the low temperature side will cause accumulation of the electrons on that side, producing an electric field which opposes the flow. At equilibrium, the effect of the electric field, $\boldsymbol{E}$, exactly cancels that of temperature gradient, ∇T. This is called the Seebeck effect. The thermopower, Q, relates $\boldsymbol{E}$ and ∇T as follows:

$$\boldsymbol{E} = \text{Q}\nabla T \tag{1.32}$$

To estimate the thermopower, note that in the above one-dimensional heat flow model the mean electron velocity at a point x due to the temperature gradient is

$$v_\text{Q} = \tfrac{1}{2}[v_x(x - v_x\tau) - v_x(x + v_x\tau)\ ] = -\tau v\frac{dv_x}{dx} = -\tau\frac{d}{dx}\left(\frac{v_x^{\,2}}{2}\right). \tag{1.33}$$

Drude Model

Write $v^2 = v_x^2/3$, we have

$$\vec{v}_Q = -\frac{\tau}{6}\frac{dv^2}{dT}\vec{\nabla}T \tag{1.34}$$

The mean velocity due to the thermoelectric field is

$$\boldsymbol{v}_E = -e\boldsymbol{E}\tau/m \tag{1.35}$$

Requiring that $\boldsymbol{v}_Q = \boldsymbol{v}_E$, we have

$$Q = -\left(\frac{1}{3e}\right)\frac{d}{dT}\frac{mv^2}{2} = -\frac{c_v}{3ne} = -\frac{k_B}{2e} = -0.43\times10^{-4}\ \text{V/K}. \tag{1.36}$$

Observed values of thermopower at room temperature are of the order of μV/K, which is 100 times smaller than Drude's prediction, illustrating inadequacy of his model.