Standing Waves

When two waves of the same frequency and amplitude travel in opposite directions in a medium, the result is a standing wave - a wave that does not travel one way or the other.

Simulation

The two waves can be represented by the equations:

\[ y_1 = A \sin(kx - \omega t) \quad \text{and} \quad y_2 = A \sin(kx + \omega t) \]

The resultant wave is their sum, and can be written as:

\[ y = 2A \sin(kx) \cos(\omega t) \]

This is quite different from the equation of a traveling wave because the spatial part is separated from the time part. It tells us that there are certain positions where the amplitude is always zero - these points are called nodes. There are other points halfway between the nodes where the amplitude is maximum - these are the anti-nodes.

Standing waves produced by reflection

We can also produce standing waves in a string by setting up periodic vibrations at one end of the string and let the wave propagate down the string and be reflected at the other end. Under special conditions, a standing wave can be produced. Those special conditions depend on the kind of reflection (fixed-end or free-end) the string is subject to.

Standing waves can also be produced by reflecting the wave at both ends.

Fixed-end Reflection

How waves reflect at the ends of a medium, or at the interface between two media, is critical to understanding things like musical instruments.

When a wave encounters a fixed end, for instance, it comes back upside down.
When a wave encounters a free end, it comes back upright.

Standing waves: a string fixed at both ends

A wave traveling in one direction on a string reflects off the end, and returns inverted because the end is fixed. This gives two identical waves traveling in opposite directions on the string, which results in a standing wave.

If the other end is also fixed, this standing wave must have zero displacement or completely constructive interference at the other end at all times. This is possible only when the wavelength is related to the length \( L \) of the string by:

\[
\frac{n \lambda}{2} = L \quad \text{where } n = 1, 2, 3, ...
\]

Using \( f = \frac{v}{\lambda} \), the corresponding frequencies are:

\[
f_n = \frac{n v}{2L}, \text{ where } n = 1, 2, 3, ...
\]

Standing waves: a string fixed at both ends

All stringed musical instruments have strings fixed at both ends. When they are played, the sound you hear is some combination of the fundamental frequency and the different harmonics - it's because the harmonics are included that the sound sounds musical. A pure sine wave does not sound nearly so nice.
Standing waves: a tube open at both ends

For a tube that opens at both ends, reflections of the sound at both ends produce a large-amplitude wave for particular resonance frequencies. For the standing waves, an open end is an anti-node (maximum amplitude point) for displacement.

Simulation: transverse representation
Simulation: longitudinal representation

The resonance frequencies are given by the same equation we used for the string:

\[ f_n = \frac{n \nu}{2L} \]

Standing waves: a tube closed at one end only

For a tube closed at one end, the closed end is a node (zero displacement) while the open end is an anti-node (maximum displacement). This leads to a different equation for the resonance frequencies.

\[ f_n = \frac{n \nu}{4L} \]

where \( n \) can only be odd integers

Simulation: transverse representation
Simulation: longitudinal representation

Two Pipes

(a) For a pipe that has only one end closed, it is found that the frequencies of two successive harmonics differ by 100 Hz. What is the length of the pipe? Take the speed of sound to be 300 m/s.

(b) Suppose you want to design a pipe that is open at both ends and has a fundamental frequency differing from that of the pipe in (a) by 20 Hz. What are the possible choices for the length of your pipe?

Standing waves: a tube closed at one end or a string fixed at one end

\[ \lambda_n = \frac{2L}{n} \] or \[ f_n = \frac{n \nu}{2L} \]

Transverse representation for the first five harmonics:

For pipes closed at one end only,

\[ \lambda_n = \frac{4L}{n} \] or \[ f_n = \frac{n \nu}{4L} \]

n = 1, 3, 5, ...

Standing waves: a tube open at both ends

Two Pipes

(a) For a pipe that has only one end closed, it is found that the frequencies of two successive harmonics differ by 100 Hz. What is the length of the pipe? Take the speed of sound to be 300 m/s.

For pipes closed at one end only,

\[ f_n = \frac{n \nu}{4L} \]

n = 1, 3, 5, ...

\[ f_{n+2} - f_n = 2 \nu / (4L) = 100 \text{ Hz} \]

\[ \Rightarrow L = \nu / (2 \times 100 \text{ Hz}) = (300 \text{ m/s}) / (200 \text{ Hz}) = 1.5 \text{ m} \]
Two Pipes

(b) Suppose you want to design a pipe that is open at both ends and has a fundamental frequency differing from that of the pipe in (a) by 20 Hz. What are the possible choices for the length of your pipe?

The fundamental frequency of the pipe in (a) is:
\[ f_1 = \frac{v}{4L} = \frac{300 \text{ m/s}}{4 \times 1.5 \text{ m}} = 50 \text{ Hz} \]

So the desired fundamental frequencies of the new pipe are:
\[ f_1 = 30 \text{ Hz} \quad \text{and} \quad f_1 = 70 \text{ Hz} \]

For pipes open at both ends,
\[ f_n = \frac{n v}{2L} \quad n = 1 \text{ for the fundamental frequency}. \]

So, \( L = \frac{v}{2 f_1} = \frac{300 \text{ m/s}}{60 \text{ Hz}} = 5 \text{ m} \)

or \( L = \frac{300 \text{ m/s}}{140 \text{ Hz}} = 2.14 \text{ m} \)

Three Pipes

(a) For a pipe that has one end closed only, it is found that the frequencies of two successive harmonics differ by \( \Delta f \). What is the length of the pipe, \( L \), in terms of \( \Delta f \) and speed of sound, \( v \)?

For pipes closed at one end,
\[ f_n = \frac{n v}{4L} \quad n = 1, 3, 5, \ldots \]

\[ \Delta f_n = f_{n+2} - f_n = 2v / (4L) \]

\[ L = \frac{v}{2 \Delta f} \]

(b) Suppose you want to design a second pipe that is open at both ends and has the same fundamental frequency as the pipe considered in (a). What should be the length of this second pipe?

The fundamental frequency of the pipe in (a) is:
\[ f_{1,a} = \frac{v}{4L} \]

Let the length of the second pipe by \( L_2 \). Then the fundamental frequency of the second pipe is:
\[ f_{1,b} = \frac{v}{2L_2} \]

\[ f_{1,a} = f_{1,b} \Rightarrow \frac{v}{4L} = \frac{v}{2L_2} \Rightarrow L_2 = 2L \]

(c) Consider a third pipe that also has one end closed only. Suppose its third harmonic has the same frequency as the first harmonic of the pipe considered in (a). What should be the length of this third pipe in terms of the length of the first pipe, \( L \)?

Given the same speed of sound, sound waves with the same frequency have the same wavelength the same. Therefore, the third harmonic frequency of the third pipe, \( f_{3,c} \), must be the same as the first harmonic frequency of the first pipe, \( f_{1,a} \). It follows that the transverse representations of the first and third harmonics of the first and third pipe are:

First pipe, \( n = 1 \):

Third pipe, \( n = 3 \):

From the transverse representations, we can see that the difference in length between the first and third pipe is one wavelength or \( 4L \). Therefore, the length of the third pipe is \( 4L + L = 5L \).

Alternatively, you can find this answer by considering the following:

For the first harmonic of the first pipe, \( \lambda_{1a}/4 = L \)

For the third harmonic of the third pipe, \( \lambda_{3c}/4 = L_3 \)

\[ \lambda_{1a} = \lambda_{3c} \Rightarrow 4L = 4L_3/5 \Rightarrow L_3 = 5L \]

Three Pipes

First pipe, \( n = 1 \):

Third pipe, \( n = 3 \):

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\[ \lambda_{1a} = \lambda_{3c} \Rightarrow 4L = 4L_3/5 \Rightarrow L_3 = 5L \]