Rotational Kinetic Energy

Rotational Kinetic Energy

Energy associated with rotation is given by an equation analogous to that for straight-line motion.

For an object that is moving but not rotating: $K = \frac{1}{2}mv^2$

For an object that is rotating only: $K = \frac{1}{2}I\omega^2$

For an object that is rolling, i.e., translating and rotating simultaneously, the total kinetic energy of such an object is:

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Racing Shapes - Revisited

We have three objects, a solid disk, a ring, and a solid sphere, all with the same mass, *M* and radius, *R*. If we release them from rest at the top of an incline, which object will win the race? Assume the objects roll down the ramp without slipping.

1. The sphere
2. The ring
3. The disk
4. It's a three-way tie
5. Can't tell - it depends on mass and/or radius.

Racing Shapes - Revisited

Question: Which shape will have the biggest velocity after rolling down the slope?

Solution: Let's use conservation of energy to analyze the race between two objects that roll without slipping down the ramp.

Let's analyze a generic object with a mass *M*, radius *R*, and a rotational inertia of:

$$I = cMR^2$$

Start with the usual five-term energy conservation equation.

$$U_i + K_i + W_{nc} = U_f + K_f$$

Eliminate the terms, K_i and U_f that are zero, we have $U_i = K_f$

Insert the expressions for the U_i and K_f.

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \qquad (1)$$

Racing Shapes - Revisited

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \qquad (1)$$

Because the object rolls without slipping, we can use $\omega = \frac{v}{R}$

Next, substitute $I = cMR^2$, where c = ½ for the disc, 2/5 for the sphere and 1 for the ring.

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}(cMR^2)\frac{v^2}{R^2}$$

Both the mass and the radius cancel out!

$$gh = \frac{1}{2}v^2 + \frac{1}{2}cv^2$$

Solving for the speed at the bottom: $v = \sqrt{\frac{2gh}{1+c}}$

What does this tell us?

$$Mgh = \underbrace{\tfrac{1}{2}Mv^2}_{\text{Translational KE}} + \underbrace{\tfrac{1}{2}cMv^2}_{\text{Rotational KE}}$$

$$v = \sqrt{\frac{2gh}{1+c}}$$

This result shows that the larger the value of *c*, the slower the object is, because a larger fraction of the potential energy is directed toward the rotational kinetic energy, with less available for the translational kinetic energy and so the object moves (translates) more slowly.

Simulation

A Figure Skater - Revisit

A spinning figure skater is an excellent example of angular momentum conservation. The skater starts spinning with her arms outstretched, and has a rotational inertia of I_i and an initial angular velocity of ω_i. When she moves her arms close to her body, she spins faster. Her moment of inertia decreases, so her angular velocity must increase to keep the angular momentum constant.

Conserving angular momentum:

$$\vec{L}_i = \vec{L}_f$$

$$I_i \vec{\omega}_i = I_f \vec{\omega}_f$$

Question: In this process, what happens to the skater's kinetic energy?

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A Figure Skater - Revisit

Question: When the figure skater moves her arms in closer to her body while she is spinning, what happens to the skater's rotational kinetic energy?

1. It increases
2. It decreases
3. It must stay the same, because of conservation of energy

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A Figure Skater - Revisit

$$K_i = \frac{1}{2} I_i \omega_i^2 = \frac{1}{2}\left(I_i \omega_i\right) \times \omega_i$$

$$K_f = \frac{1}{2} I_f \omega_f^2 = \frac{1}{2}\left(I_f \omega_f\right) \times \omega_f$$

The terms in brackets are the same, so the final kinetic energy is larger than the initial kinetic energy, because $\omega_i < \omega_f$.

Where does the extra kinetic energy come from?

The skater does work on her arms in bringing them closer to her body, and that work shows up as an increase in kinetic energy.

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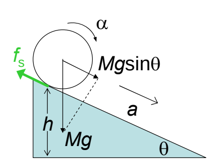
A ball rolling down a ramp

Question: A ball with mass M and radius R rolls without slipping down a ramp from the top to the bottom (see figure). We have found that $a = g\sin\theta/(1 + c)$ and $f_s = Mg\sin\theta/(1/c + 1)$, where $c = 2/5$. Use conservation of mechanical energy to find the non-conservative work done, W_{nc}, on the ball when it reaches the bottom. Assume that the ball is initially at rest and at a height h above ground.



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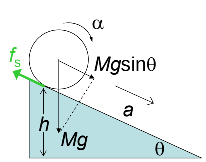
A ball rolling down a ramp

Solution:

Conservation of mechanical energy gives $E_i + W_{nc} = E_f$



Initial mechanical energy,
$E_i = K_i + U_i = 0 + Mgh = Mgh$

Final mechanical energy,
$E_f = K_f + U_f$
$= ½ mv_f^2 + ½ I\omega_f^2 + ½ Mv_f^2 + (c/2)Mv_f^2$
$= (1+c)Mv_f^2/2$

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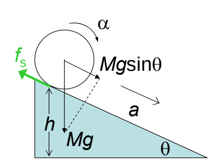
A ball rolling down a ramp

To find v_f, we use

$v_f^2 = v_i^2 + 2as$
$= 0 + 2[-g\sin\theta/(1+c)][-h/\sin\theta]$
$= 2gh/(1+c)$



$E_i + W_{nc} = Mgh - W_{nc}$

$E_f = (1+c)Mv_f^2/2 = Mgh$

$E_i + W_{nc} = E_f$ gives $W_{nc} = 0$

This shows that f_s actually does no work on the ball!

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