Applications of Newton’s Second Law II and III

Atwood’s Machine

If $a = 0$ (at equilibrium),

$$ F = (M - m)g, \quad T = (M - m)g $$

If $F > 0$ (M accelerates downward while $m$ accelerates upward,

$$ a = (M - m)g/(M + m) \quad T = mg + a $$

Motion of Two Boxes Connected by a String

Consider two boxes with masses $m_1$ and $m_2$ connected by a string as shown in the diagram at right. Suppose the coefficient of kinetic friction between the $m_1$ box and the table is $\mu_k$. What is the acceleration of the two boxes? What is the tension in the string?

Ans.

$$ a = \frac{m_2g - \mu_k m_1 g}{m_2 + m_1} = \frac{(m_2 - m_1)g}{m_2 + m_1} \cdot \text{friction} \quad T = m_2(g - a) $$

Motion of Three Boxes Connected by Two Strings -- No Friction

Suppose there’s no friction between the boxes A, B, C and the table and $F_C > F_A$. What’s the acceleration of the boxes and tensions $T_{AB}$ and $T_{BC}$?

To solve the problem, the usual way is to write the Newton’s equations: $T_{AB} - F_A = m_Aa$ (for block A), $T_{BC} - T_{AB} = m_Ba$ (for block B), and $F_C - T_{BC} = m_Ca$ (for block C).

Then solve these equations to find $a$, $T_{AB}$ and $T_{BC}$. But a much simpler way is to consider the net tangential force (i.e., tangential to the strings) acting on the system as a whole. In this case, the tangential net force, $F_{net} = F_C - F_A$.

So, $a = F_{net}/(\text{Total mass}) = (F_C - F_A)/(m_A + m_B + m_C)$

To find $T_{AB}$, use the Newton’s equation for block A:

$$ T_{AB} - F_A = m_Aa \quad \Rightarrow \quad T_{AB} = F_A + m_Aa = F_A + m_A(F_C - F_A)/(m_A + m_B + m_C) $$

To find $T_{BC}$, use the Newton’s equation for block C:

$$ F_C - T_{BC} = m_Ca \quad \Rightarrow \quad T_{BC} = F_C - m_Ca = F_C - m_C(F_C - F_A)/(m_A + m_B + m_C) $$
Motion of Three Boxes Connected by Two Strings  
-- with Friction

Suppose the coefficient of kinetic friction between the boxes A, B, C and the table is $\mu_k$ and $F_C > F_A$. What’s the acceleration of the boxes and tensions $T_{AB}$ and $T_{BC}$?

The net force $F_{net}$ acting on the system as a whole is $F_C - F_A - f_{k,tot}$. Note that the tensional forces do not contribute to the net force of the system because they come in pairs of equal magnitude and opposite direction and so cancel exactly among themselves.

For the system as a whole, $f_{k,tot} = \mu_k(m_A + m_B + m_C)g$.

So, $a = F_{net}/(Total\ mass) = (F_C - F_A)/(m_A + m_B + m_C) - \mu_k g$.

To find $T_{BC}$, write the Newton’s equation for block C:

$$T_{BC} = F_C - m_C[(F_C - F_A)/(m_A + m_B + m_C) - \mu_k g]$$

This is the same tension we found when there’s no friction!

To find $T_{AB}$, write the Newton’s equation for block A:

$$T_{AB} = F_A + m_Aa + f_{kA} = F_A + m_A[(F_C - F_A)/(m_A + m_B + m_C) - \mu_k g]$$

Again, this is the same tension we found when there’s no friction.

Motion of Two Boxes Connected by a String  
-- No friction

Consider the situation shown at right, where the pulley is frictionless and massless. Find the acceleration $a$ and tension $T$ in terms of $m_1$, $m_2$ and $g$, if there’s no friction between $m_1$ and the incline.

First, draw the FBD of $m_1$ and $m_2$:

Neglect friction

From the FBD of $m_2$, we can see that the tangential force, besides tension $T$, that’s acting on $m_2$ is $m_2 g \sin \theta$. Similarly, from the FBD of $m_1$, the tangential force besides $T$, that’s acting on $m_1$ is $m_1 g$. We arbitrarily assume that $m_1$ weighs on the system more than $m_2$ does. (If this is a wrong guess, the sign of $a$ we get will be negative, which will correct for the wrong direction.) With this, the net tangential force acting on the system as a whole can be written as:

$$F_{net} = m_1 g - m_2 g \sin \theta$$

So, $a = \frac{m_1 g - m_2 g \sin \theta}{m_1 + m_2}$.
Motion of Two Boxes Connected by a String
-- No friction

Next, use the FBD of \( m_1 \) to find \( T \):

\[
T = m_1(g - a)
\]

Neglect friction

\[
T = m_1g - m_1 \sin \theta \frac{m_1g (1 + \sin \theta)}{m_1 + m_2}
\]

Consider the FBD of \( m_1 \):

\[
T = m_1a = m_1m_2g \sin \theta \frac{1}{m_1 + m_2}
\]

Friction on \( m_1 \) but none on \( m_2 \).
Motion of Two Boxes Connected by a String (4) -- with Friction

The net tangential force acting on the system as a whole is:

\[ F_{\text{net}} = m_2 g \sin \theta - f_k \]

So, \( a = \frac{F_{\text{net}}}{\text{(total mass)}} \) = \( \frac{(m_2 g \sin \theta - f_k)}{(m_1 + m_2)} \)

Next, substitute \( f_k = \mu m_1 g \). We get:

\[ a = \frac{(m_2 g \sin \theta - \mu m_1 g)}{(m_1 + m_2)} \]

From last page,

\[ T - f_k = m_1 a_0 - \mu m_1 m_2 g \( / \( m_1 + m_2 \) \) \]

Next, substitute \( f_k = \mu m_1 g \):

\[ T = m_1 a_0 + \mu m_1 m_2 g \( / \( m_1 + m_2 \) \) \]

\[ = \text{value of tension when friction is zero. This result shows that tension is increased when friction is turned on. This is because the amount of acceleration reduction is less than accountable by the frictional force, i.e., } m_1 \Delta a < f_k \text{. So, tension has to work harder than before to overcome friction while providing the correct acceleration consistent with } a = \frac{F_{\text{net}}}{\text{(total mass)}}. \]

Motion of Three Boxes Connected by Two Strings (1) -- no friction

Suppose there’s no friction between box B and the table. Find the acceleration of the system and tensions \( T_1 \) and \( T_2 \). You may also assume that the pulleys are frictionless and massless.

Let’s arbitrarily assume that block A weighs more than block C. Then the net force acting on the system is \((m_c - m_A)g\)

So, \( a = \frac{F_{\text{net}}}{\text{(Total mass)}} = \frac{(m_c - m_A)g}{(m_A + m_B + m_C)} \)

To find \( T_1 \), write the Newton’s equation for block A:

\[ T_1 - m_A g = m_A a \]

\[ \Rightarrow T_1 = m_A (g + a) \]

Substitute \( a = \frac{(m_c - m_A)g}{(m_A + m_B + m_C)} \), we get:

\[ T_1 = m_A (m_B + 2m_C)g \( / \( m_A + m_B + m_C \) \) \]

To find \( T_2 \), write the Newton’s equation for block C:

\[ T_2 - m_C g = -m_C a \]

\[ \Rightarrow T_2 = m_C (g - a) \]

Substitute \( a = \frac{(m_c - m_A)g}{(m_A + m_B + m_C)} \), we get:

\[ T_2 = m_C (2m_A + m_B)g \( / \( m_A + m_B + m_C \) \) \]
For the system considered in the last example, suppose there's friction between box B and the table. Find $a$, $T_1$, and $T_2$. Assume that the coefficient of kinetic friction is $\mu_k$.

Let's still assume that block A weighs more than block C. Then the net force acting on the system is $(m_c - m_A)g - f_k = (m_c - m_A)g - \mu km_B g$

So, $a = \frac{\text{Net force}}{\text{Total mass}} = \frac{(m_c - m_A - \mu km_B)g}{m_A + m_B + m_C}$

To find $T_1$, write the Newton's equation for block A:

$$T_1 - m_A g = m_A a$$

$$\Rightarrow T_1 = m_A(g + a)$$

Note that the value of $T_1$ without tension is $T_{1,0} = m_A(g + a_0)$, where $a_0$ is the acceleration of the system when there’s no friction. Clearly, $a = a_0 - \frac{\mu km_B g}{(\Sigma m)}$. So,

$$T_1 = T_{1,0} - \frac{\mu km_B m_A g}{(\Sigma m)}$$

Substitute $T_{1,0} = m_A[2m_A + m_B(1+\mu_k)] / (m_A + m_B + m_C)$, we get:

$$T_1 = m_A g[2m_A + m_B(1+\mu_k)] / (m_A + m_B + m_C)$$

To find $T_2$, write the Newton's equation for block C:

$$T_2 - m_C g = -m_C a$$

$$\Rightarrow T_2 = m_C(g - a)$$

Note that the value of $T_2$ without tension is $T_{2,0} = m_C(g + a_0)$, where $a_0$ is the acceleration of the system when there’s no friction. Clearly, $a = a_0 - \frac{\mu km_B g}{(\Sigma m)}$. So,

$$T_2 = T_{2,0} + \frac{\mu km_B m_C g}{(\Sigma m)}$$

Substitute $T_{2,0} = m_C[2m_A + m_B(1+\mu_k)] / (m_A + m_B + m_C)$, we get:

$$T_2 = m_C g[2m_A + m_B(1+\mu_k)] / (m_A + m_B + m_C)$$