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Local basis-dependent noise-induced Bell-nonlocality sudden death in tripartite systems

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Abstract

We demonstrate that multipartite Bell-inequality violations can be fully destroyed in finite time in three-qubit systems subject only to the mechanism of local external asymptotic dephasing noise. This broadens the study of local-noise-induced sudden death of nonlocal behavior, extending it beyond the realm of bipartite systems, to which it had previously been restricted.
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1. Introduction

It has been increasingly recognized that the rates of loss of joint-state coherence and of entanglement, two fundamental characteristics of quantum states, may differ within a composite system subject to local external noise [1–10]. Moreover, it has been shown that for some classes of states entanglement sudden death (ESD) [8], the disentanglement of bipartite systems in *finite time* subject only to the mechanism of basis-dependent local phase noise, occurs. Thus, qualitative as well as quantitative differences in coherence and nonlocality have been demonstrated [4–10]. Here, this investigation is further advanced.

Previously, such differences of noise-induced behavior have been carefully explored only in bipartite systems. The study of ESD under local dephasing noise has not been demonstrated in any multipartite system of more than two components because properly quantifying multipartite entanglement, particularly for mixed states which it involves by necessity, is problematic for systems of more than two qudits [11,12]. Nonetheless, as we demonstrate here, one may still demonstrate the existence of local-noise induced death of nonlocal behavior with tools currently at hand. In particular, one can find classes of states in which generalized Bell-nonlocality in multipartite systems can go to zero in finite time while state coherence continues to be maintained for all finite times, an effect which we term *Bell-nonlocality sudden death* (*BNSD*). Here, BNSD is demonstrated in the tripartite context: the destruction of nonlocality as measured by the extent of violation of tripartite Bell inequalities in finite time under basis-dependent multi-local asymptotic dephasing noise is demonstrated in a class of initially Bell-nonlocal pure states of three-qubit systems, namely, the W class of states.

The extension of the consideration of the sudden death of nonlocal properties due to local dephasing noise beyond the bipartite case to the tripartite case is important because tripartite systems can exhibit fundamental characteristics impossible in bipartite sys-

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tems, even at the three-qubit level, for example, see [13,14]. The exhibition of BNSD illuminates the quantum–classical transition, quantum measurement, and quantum information processing where joint-state coherence and nonlocality are typically considered crucial. Our demonstration of BNSD shows that quantum information processing may be even more challenging to carry out in a noisy environment than previously thought.

2. Model: Initial state and noise model

There are two distinct classes of entangled pure states for three-qubit systems, the GHZ class and the W class, each represented by a characteristic state related to all others of their class by stochastic local operations and classical communication [15].[1] Here, we restrict our attention to systems initially prepared in pure states of the W class in order to show that tripartite Bell-inequality violations in such systems can be eliminated by the mechanical influence of local asymptotic dephasing noise alone.

The general pure W state is given by $|\mathrm{W_g}\rangle = \bar{a}_1|001\rangle + \bar{a}_2|010\rangle + \bar{a}_4|100\rangle$, where $\bar{a}_i \in \mathcal{C}$ and $\sum_i |\bar{a}_i|^2 = 1$, and has the associated density matrix

$$\rho_{\mathrm{W_g}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & |\bar{a}_1|^2 & \bar{a}_1\bar{a}_2^* & 0 & \bar{a}_1\bar{a}_4^* & 0 & 0 & 0 \\ 0 & \bar{a}_2\bar{a}_1^* & |\bar{a}_2|^2 & 0 & \bar{a}_2\bar{a}_4^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{a}_4\bar{a}_1^* & \bar{a}_4\bar{a}_2^* & 0 & |\bar{a}_4|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

In order to demonstrate the existence of BNSD in multipartite systems, we model pure dephasing noise acting locally on each of the three qubits which are assumed to be isolated from each other. The most general time-evolved open-system density matrix expressible in the operator-sum decomposition is

$$\rho(t) = \mathcal{E}\big(\rho(0)\big) = \sum_{\mu} D_\mu(t)\rho(0)D_\mu^\dagger(t), \tag{2}$$

where the $D_\mu(t)$, which satisfy a completeness condition guaranteeing that the evolution be trace-preserving, represent the influence of local statistical noise, and where the index μ runs over the number of elements required for the decomposition [17]. For local and multi-local dephasing environments, the $D_\mu(t)$ are of the form $G_k(t)F_j(t)E_i(t)$, so that

$$\rho(t) = \mathcal{E}\big(\rho(0)\big) = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} G_k(t)F_j(t)E_i(t)\rho(0)E_i^\dagger(t)F_j^\dagger(t)G_k^\dagger(t),$$

where

$$E_1(t) = \mathrm{diag}\big(1, \gamma_{\mathrm{A}}(t)\big) \otimes \mathbf{I} \otimes \mathbf{I}, \qquad E_2(t) = \mathrm{diag}\big(0, \omega_{\mathrm{A}}(t)\big) \otimes \mathbf{I} \otimes \mathbf{I}, \tag{3}$$

$$F_1(t) = \mathbf{I} \otimes \mathrm{diag}\big(1, \gamma_{\mathrm{B}}(t)\big) \otimes \mathbf{I}, \qquad F_2(t) = \mathbf{I} \otimes \mathrm{diag}\big(0, \omega_{\mathrm{B}}(t)\big) \otimes \mathbf{I}, \tag{4}$$

$$G_1(t) = \mathbf{I} \otimes \mathbf{I} \otimes \mathrm{diag}\big(1, \gamma_{\mathrm{C}}(t)\big), \qquad G_2(t) = \mathbf{I} \otimes \mathbf{I} \otimes \mathrm{diag}\big(0, \omega_{\mathrm{C}}(t)\big), \tag{5}$$

$\gamma_{\mathrm{A}}(t) = \gamma_{\mathrm{B}}(t) = \gamma_{\mathrm{C}}(t) = \gamma(t) = e^{-\Gamma t}$, $\omega_{\mathrm{A}}(t) = \omega_{\mathrm{B}}(t) = \omega_{\mathrm{C}}(t) = \omega(t) = \sqrt{1-\gamma^2(t)} = \sqrt{1-e^{-2\Gamma t}}$. The $E_i(t)$, $F_j(t)$, and $G_k(t)$ induce dephasing in the state of each qubit alone, and individually satisfy the usual completeness condition for the operator-sum decomposition of CPTP maps [17]. Γ quantifies the rate of local asymptotic dephasing which we assume to be equal for all qubits. We take the time-dependence of $\gamma(t)$'s to be implicit from here on, particularly when explicitly displaying the forms of density matrices.

[1] Illuminating results for Bell-type inequalities in the case of GHZ states can be found, for example, in [16].

3. Bell-nonlocality sudden death

In the multi-local noise environment described in the previous section, when a three-qubit system is prepared at the initial time $t = 0$ in the generic pure W state $\rho(0) = |\mathrm{W_g}\rangle\langle \mathrm{W_g}|$, the time-evolved state $\rho(t)$, that is, the solution of Eq. (2) is

$$\rho(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & |\bar{a}_1|^2 & \bar{a}_1\bar{a}_2^*\gamma_\mathrm{B}\gamma_\mathrm{C} & 0 & \bar{a}_1\bar{a}_4^*\gamma_\mathrm{A}\gamma_\mathrm{C} & 0 & 0 & 0 \\ 0 & \bar{a}_2\bar{a}_1^*\gamma_\mathrm{B}\gamma_\mathrm{C} & |\bar{a}_2|^2 & 0 & \bar{a}_2\bar{a}_4^*\gamma_\mathrm{A}\gamma_\mathrm{B} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{a}_4\bar{a}_1^*\gamma_\mathrm{A}\gamma_\mathrm{C} & \bar{a}_4\bar{a}_2^*\gamma_\mathrm{A}\gamma_\mathrm{B} & 0 & |\bar{a}_4|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

The off-diagonal elements of this matrix undergo a simple exponential decay, the three-qubit state fully decohering only in the infinite-time limit, wherein the γ factors approach zero. As we now show, however, the tripartite Bell nonlocality of states is entirely lost in a specific finite time-scale.

For N-qubit systems, there are two multipartite Bell inequalities that have been used to detect the degree of nonlocality as measured by the extent of their violation. Satisfaction of the first, that of Zukowski and Brukner [18], is both a necessary and a sufficient condition for a quantum system of N qubits obey Bell locality but is often difficult computationally to determine; it is currently unclear whether a calculation valid for all classes of states for $N > 3$ can be carried out. The second inequality used for this purpose is the Mermin–Ardehali–Belinksii–Klyshko (MABK) inequality [19–21], which may fail to detect the existence of nonlocal correlation in some states for cases where $N > 3$ but always detects nonlocal correlation when $N = 3$, has the advantage of being readily computable. Because these two inequalities are equivalent in the case $N = 3$, we have a definitive criterion for Bell locality, the extent of the violation of which allows us to show its death, that is valid for all three-qubit states and so for the case of W states.

A quantum state ρ violates the MABK inequality if

$$\left|\langle \mathcal{B}_N\rangle_\rho\right| > 1. \tag{7}$$

In the simplest case of two qubits, $N = 2$, the operator $\mathcal{B}_N$ on the left-hand side of the MABK inequality is just the Clauser–Horne–Shimony–Holt (CHSH) operator [22], namely,

$$\mathcal{B}_2 = \frac{1}{2}\left[M_\mathrm{A}M_\mathrm{B} + M_\mathrm{A}M'_\mathrm{B} + M'_\mathrm{A}M_\mathrm{B} - M'_\mathrm{A}M'_\mathrm{B}\right]. \tag{8}$$

In the case of current interest, that of three qubits (here labeled K = A, B, or C), the specific form of the operator $\mathcal{B}_N$ is

$$\mathcal{B}_3 = \frac{1}{2}\left[M_\mathrm{A}M_\mathrm{B}M'_\mathrm{C} + M_\mathrm{A}M'_\mathrm{B}M_\mathrm{C} + M'_\mathrm{A}M_\mathrm{B}M_\mathrm{C} - M'_\mathrm{A}M'_\mathrm{B}M'_\mathrm{C}\right], \tag{9}$$

the primed and unprimed terms denoting the two different directions in which the corresponding party measures; the measurement operators M_K and M'_K, denoting the operators corresponding to measurements on the qubit K, where the second corresponds to a measurement performed along a direction differing by θ relative that performed on the first qubit, are

$$\begin{pmatrix} M_\mathrm{K} \\ M'_\mathrm{K} \end{pmatrix} = R(\theta_\mathrm{K})\begin{pmatrix} M_\mathrm{A} \\ M'_\mathrm{A} \end{pmatrix}, \tag{10}$$

where $R(\theta_\mathrm{K}) = \left(\begin{smallmatrix} \cos\theta_\mathrm{K} & -\sin\theta_\mathrm{K} \\ \sin\theta_\mathrm{K} & \cos\theta_\mathrm{K} \end{smallmatrix}\right)$.

In the case of three qubits, the relative angles of measurement are $\theta_\mathrm{B} = \pi/6$ and $\theta_\mathrm{C} = \pi/3$. Thus, the corresponding measurement operators for qubits A, B, and C are written in terms of the usual Pauli operators σ_z and σ_x, as

$$M_\mathrm{A} = \sigma_z \otimes \mathbf{I} \otimes \mathbf{I}, \tag{11}$$

$$M'_\mathrm{A} = \sigma_x \otimes \mathbf{I} \otimes \mathbf{I}, \tag{12}$$

$$M_\mathrm{B} = \mathbf{I} \otimes \left[\cos\left(\frac{\pi}{6}\right)\sigma_z - \sin\left(\frac{\pi}{6}\right)\sigma_x\right] \otimes \mathbf{I}, \tag{13}$$

$$M'_\mathrm{B} = \mathbf{I} \otimes \left[\sin\left(\frac{\pi}{6}\right)\sigma_z + \cos\left(\frac{\pi}{6}\right)\sigma_x\right] \otimes \mathbf{I}, \tag{14}$$

$$M_\mathrm{C} = \mathbf{I} \otimes \mathbf{I} \otimes \left[\cos\left(\frac{\pi}{3}\right)\sigma_z - \sin\left(\frac{\pi}{3}\right)\sigma_x\right], \tag{15}$$

$$M'_{\mathrm{C}} = \mathbf{I} \otimes \mathbf{I} \otimes \left[\sin\left(\frac{\pi}{3}\right)\sigma_z + \cos\left(\frac{\pi}{3}\right)\sigma_x\right]. \tag{16}$$

The expectation value of the $\mathcal{B}_3$ operator for the state under the influence of multi-local noise on three qubits initially prepared in the generic pure W state is

$$\begin{aligned}\langle\mathcal{B}_3\rangle_{\rho(t)} &= \mathrm{tr}\big[\mathcal{B}_3(t)\rho(t)\big] \\ &= \mathrm{tr}\left[\frac{1}{2}\big(M_{\mathrm{A}}M_{\mathrm{B}}M'_{\mathrm{C}} + M_{\mathrm{A}}M'_{\mathrm{B}}M_{\mathrm{C}} + M'_{\mathrm{A}}M_{\mathrm{B}}M_{\mathrm{C}} - M'_{\mathrm{A}}M'_{\mathrm{B}}M'_{\mathrm{C}}\big)\rho(t)\right] \\ &= \frac{1}{2}\,\mathrm{tr}\big[M_{\mathrm{A}}M_{\mathrm{B}}M'_{\mathrm{C}}\rho(t)\big] + \frac{1}{2}\,\mathrm{tr}\big[M_{\mathrm{A}}M'_{\mathrm{B}}M_{\mathrm{C}}\rho(t)\big] + \frac{1}{2}\,\mathrm{tr}\big[M'_{\mathrm{A}}M_{\mathrm{B}}M_{\mathrm{C}}\rho(t)\big] - \frac{1}{2}\,\mathrm{tr}\big[M'_{\mathrm{A}}M'_{\mathrm{B}}M'_{\mathrm{C}}\rho(t)\big].\end{aligned} \tag{17}$$

Therefore, one finds

$$\frac{1}{2}\,\mathrm{tr}\big[M_{\mathrm{A}}M_{\mathrm{B}}M'_{\mathrm{C}}\rho(t)\big] = -\frac{1}{8}\big[3 + \big(\bar{a}_1\bar{a}_2^* + \bar{a}_2\bar{a}_1^*\big)\gamma_{\mathrm{B}}\gamma_{\mathrm{C}}\big], \tag{18}$$

$$\frac{1}{2}\,\mathrm{tr}\big[M_{\mathrm{A}}M'_{\mathrm{B}}M_{\mathrm{C}}\rho(t)\big] = -\frac{1}{8}\big[1 + 3\big(\bar{a}_1\bar{a}_2^* + \bar{a}_2\bar{a}_1^*\big)\gamma_{\mathrm{B}}\gamma_{\mathrm{C}}\big], \tag{19}$$

$$\frac{1}{2}\,\mathrm{tr}\big[M'_{\mathrm{A}}M_{\mathrm{B}}M_{\mathrm{C}}\rho(t)\big] = -\frac{1}{8}\gamma_{\mathrm{A}}\big[\big(\bar{a}_2\bar{a}_4^* + \bar{a}_4\bar{a}_2^*\big)\gamma_{\mathrm{B}} + 3\big(\bar{a}_1\bar{a}_4^* + \bar{a}_4\bar{a}_1^*\big)\gamma_{\mathrm{C}}\big], \tag{20}$$

$$\frac{1}{2}\,\mathrm{tr}\big[M'_{\mathrm{A}}M'_{\mathrm{B}}M'_{\mathrm{C}}\rho(t)\big] = \frac{1}{8}\gamma_{\mathrm{A}}\big[3\big(\bar{a}_2\bar{a}_4^* + \bar{a}_4\bar{a}_2^*\big)\gamma_{\mathrm{B}} + \big(\bar{a}_1\bar{a}_4^* + \bar{a}_4\bar{a}_1^*\big)\gamma_{\mathrm{C}}\big]. \tag{21}$$

We then have

$$\big|\langle\mathcal{B}_3\rangle_{\rho(t)}\big| = \frac{1}{2}\big|1 + \big(\bar{a}_1\bar{a}_2^* + \bar{a}_2\bar{a}_1^*\big)\gamma_{\mathrm{B}}\gamma_{\mathrm{C}} + \big(\bar{a}_2\bar{a}_4^* + \bar{a}_4\bar{a}_2^*\big)\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} + \big(\bar{a}_1\bar{a}_4^* + \bar{a}_4\bar{a}_1^*\big)\gamma_{\mathrm{A}}\gamma_{\mathrm{C}}\big|. \tag{22}$$

Thus, tripartite Bell nonlocality is nonexistent by the time-scale

$$\tau_{\mathrm{BNSD}} = \frac{\ln(\bar{a}_1\bar{a}_2^* + \bar{a}_2\bar{a}_1^* + \bar{a}_2\bar{a}_4^* + \bar{a}_4\bar{a}_2^* + \bar{a}_1\bar{a}_4^* + \bar{a}_4\bar{a}_1^*)}{2\Gamma} \tag{23}$$

specified by the time for which $|\langle\mathcal{B}_3\rangle_{\rho(t)}|$ reaches 1 from above. This suffices to demonstrate tripartite Bell-nonlocality sudden death of initially Bell-nonlocal states under local dephasing noise: by Eq. (23), whenever $1/2 < a_1a_2 + a_2a_4 + a_1a_4$ we have $|\langle\mathcal{B}_3\rangle_{\mathrm{W_g}}| > 1$, a violation of the MABK inequality, and in the limit $t \to \infty |\langle\mathcal{B}_3\rangle_{\rho(t)}| \to 1/2$, no longer violating the inequality. In particular, consider the particular case in which all amplitudes are equal and real, that is, the standard W state. Then we find that both $a_1a_2 + a_2a_4 + a_1a_4 = 1$ and

$$\tau_{\mathrm{BNSD}} = \frac{\ln(2)}{2\Gamma}. \tag{24}$$

4. Conclusions

We have demonstrated the destruction of Bell-nonlocal behavior under basis-dependent local asymptotic dephasing noise, as measured by the extent of violation of a tripartite Bell inequality, in finite time while state coherence remains for all finite times in a class of initial states of three-qubit systems. This illuminates the quantum–classical transition, and quantum information processing in particular, because it shows in the multipartite context that nonlocal behavior can be lost simply due to the influence of local asymptotic dephasing noise, that in which practical quantum computing and quantum measurement will typically take place, despite the persistence of simple coherence for all finite times, strengthening the view that environmental noise makes quantum information processing a particularly challenging task.

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