Problem Set 2: More Linear Algebra

1 Index Notation 3 (10 points)

Following on from last week's problem, consider an $n \times n$ matrix X with entries X_{ij}, each being an independent variable that can be varied. Suppose X is invertible. A useful expression for its inverse is (assume it is a matrix with real entries, so we don't have to care about index positions)

$$(X^{-1})_{ij} = \frac{1}{\det(X)} \frac{1}{(n-1)!} \epsilon_{ji_2 \cdots i_n} \epsilon_{ik_2 \cdots k_n} X_{i_2 k_2} \cdots X_{i_n k_n} \,. \tag{1}$$

(a) Show that the expression above is correct (you may need to look up some general properties of the n-dimensional Levi-Civita symbol).

SOLUTION: Let

$$A_{ij} = \frac{1}{\det(X)} \frac{1}{(n-1)!} \epsilon_{ji_2 \cdots i_n} \epsilon_{ik_2 \cdots k_n} X_{i_2 k_2} \cdots X_{i_n k_n} \,, \tag{2}$$

Let's multiply both sides by X_{jm}. This gives

$$\begin{aligned} A_{ij} X_{jm} &= \frac{1}{\det(X)} \frac{1}{(n-1)!} \epsilon_{ji_2 \cdots i_n} \epsilon_{ik_2 \cdots k_n} X_{jm} X_{i_2 k_2} \cdots X_{i_n k_n} \\ &= \frac{1}{\det(X)} \frac{1}{(n-1)!} \epsilon_{ik_2 \cdots k_n} \epsilon_{mk_2 \cdots k_n} \det(X) \\ &= \frac{1}{(n-1)!} \cdot (n-1)! \delta^i_m \\ &= \delta^i_m \,, \end{aligned} \tag{3}$$

where we have used the identity

$$\epsilon_{ik_2 \cdots k_n} \epsilon_{mk_2 \cdots k_n} = (n-1)! \delta^i_m \,. \tag{4}$$

We have therefore shown that A_{ij} is a left-inverse of X_{jm}. By a similar argument we can also show that $X_{mj} A_{ji} = \delta_{mi}$, showing that A_{ij} is indeed the inverse of X_{ij}.

(b) Use this and the definition of the determinant to show that

$$\frac{\partial}{\partial X_{ij}} \det(X) = \det(X) (X^{-1})_{ji} \,. \tag{5}$$

SOLUTION: The expression for the determinant is

$$\det(X) = \frac{1}{n!} \epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_n} X_{i_1 j_1} \cdots X_{i_n j_n} \,. \tag{6}$$

Taking the derivative, we find

$$\begin{aligned} \frac{\partial}{\partial X_{ij}} \det(X) =& \frac{1}{n!} \epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_n} \left(\delta_{i_1 i} \delta_{j_1 j} X_{i_2 j_2} \cdots X_{i_n j_n} + X_{i_1 j_1} \delta_{i_2 i} \delta_{j_2 j} \cdots X_{i_n j_n} + \cdots \right. \\ & \left. + X_{i_1 j_1} \cdots X_{i_{n-1} j_{n-1}} \delta_{i_n i} \delta_{j_n j} \right) \\ =& \frac{1}{(n-1)!} \epsilon_{ii_2 \cdots i_n} \epsilon_{jj_2 \cdots j_n} X_{i_2 j_2} \cdots X_{i_n j_n} \,, \end{aligned} \tag{7}$$

where the last line follows from symmetry: for each of the terms, after contracting with the

Levi-Civita symbols to get e.g. $\epsilon_{i_1,\cdots,i_{p-1},i,i_{p+1},\cdots,i_n}\epsilon_{j_1,\cdots,j_{p-1},j,j_{p+1},\cdots,j_n}$, we can move both i and j to the front by the same number of swaps for each Levi-Civita symbol, and so the net effect of all of the swapping is $+1$. Comparing this with our expression for the inverse, we find

$$\frac{\partial}{\partial X_{ij}}\det(X) = \det(X)(X^{-1})_{ji} \tag{8}$$

as required.

2 (SG 10.15) Symmetric Integration (15 points)

Show that the n-dimensional integral of

$$I_{\alpha\beta\gamma\delta} = \int \frac{d^n k}{(2\pi)^n}\, k_\alpha k_\beta k_\gamma k_\delta f(k^2) \tag{9}$$

is given by

$$I_{\alpha\beta\gamma\delta} = A(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})\,, \tag{10}$$

where

$$A = \frac{1}{n(n+2)}\int \frac{d^n k}{(2\pi)^n} k^4 f(k^2)\,. \tag{11}$$

Here, $\vec{k}$ is a regular vector in $\mathbb{R}^n$.

Similarly evaluate

$$I_{\alpha\beta\gamma\delta\epsilon} = \int \frac{d^n k}{(2\pi)^n}\, k_\alpha k_\beta k_\gamma k_\delta k_\epsilon f(k^2)\,. \tag{12}$$

SOLUTION: To solve this, we want to show that $I_{\alpha\beta\gamma\delta}$ is invariant under $O(n)$ transformations. Under an orthogonal transformation O, we see that

$$I_{\alpha\beta\gamma\delta} \mapsto O^\alpha{}_{\alpha'}O^\beta{}_{\beta'}O^\gamma{}_{\gamma'}O^\delta{}_{\delta'}I_{\alpha\beta\gamma\delta} = \int \frac{d^n k}{(2\pi)^n} O^\alpha{}_{\alpha'}k_\alpha O^\beta{}_{\beta'}k_\beta O^\gamma{}_{\gamma'}k_\gamma O^\delta{}_{\delta'}k_\delta f(k^2)\,. \tag{13}$$

Let's relabel $\tilde{k}_{\alpha'} = O^\alpha{}_{\alpha'}k_\alpha$, noting that $\tilde{k}^2 = k^2$, since the transformation is orthogonal. We can also perform a change of variables in the integral, with

$$d^n k = \left|\frac{\partial \vec{k}}{\partial \tilde{\vec{k}}}\right| d^n \tilde{k}\,, \tag{14}$$

where $\left|\partial\vec{k}/\partial\tilde{\vec{k}}\right|$ is the absolute value of the Jacobian associated with the transformation. Since $k_\alpha = (O^{-1})^{\alpha'}{}_\alpha \tilde{k}_{\alpha'}$, and O^{-1} is orthogonal, we have $d^n k = d^n\tilde{k}$. Putting this altogether, we find

$$I_{\alpha\beta\gamma\delta} \mapsto \int \frac{d^n\tilde{k}}{(2\pi)^n}\tilde{k}_\alpha\tilde{k}_\beta\tilde{k}_\gamma\tilde{k}_\delta f(\tilde{k}^2) = I_{\alpha\beta\gamma\delta}\,, \tag{15}$$

since we can simply relabel again $\tilde{k} \to k$.

We therefore conclude that $I_{\alpha\beta\gamma\delta}$ is invariant under $O(n)$, and can be written as

$$I_{\alpha\beta\gamma\delta} = A\delta_{\alpha\beta}\delta_{\gamma\delta} + B\delta_{\alpha\gamma}\delta_{\beta\delta} + C\delta_{\alpha\delta}\delta_{\beta\gamma}\,. \tag{16}$$

First, note that under the swap of any index, e.g. $\alpha \leftrightarrow \beta$, $I_{\alpha\beta\gamma\delta} = I_{\beta\alpha\gamma\delta}$, and so symmetry enforces $A = B = C$. Now, let's contract α and β, as well as γ and δ. We get

$$\begin{aligned} I^{\alpha}{}_{\alpha}{}^{\gamma}{}_{\gamma} = \delta^{\alpha\beta}\delta^{\gamma\delta} I_{\alpha\beta\gamma\delta} &= A\delta^{\alpha\beta}\delta^{\gamma\delta}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \\ &= A(\delta^{\alpha\beta}\delta_{\alpha\beta}\delta^{\gamma\delta}\delta_{\gamma\delta} + \delta^{\alpha\beta}\delta_{\alpha\gamma}\delta^{\gamma\delta}\delta_{\beta\delta} + \delta^{\alpha\beta}\delta_{\alpha\delta}\delta^{\gamma\delta}\delta_{\beta\gamma}) \\ &= A(n^2 + \delta^{\beta}_{\gamma}\delta^{\gamma}_{\beta} + \delta^{\beta}_{\delta}\delta^{\delta}_{\beta}) \\ &= A(n^2 + n + n) = An(n+2)\,. \end{aligned} \tag{17}$$

On the other hand,

$$I^{\alpha}{}_{\alpha}{}^{\gamma}{}_{\gamma} = \int \frac{d^n k}{(2\pi)^n} k^4 f(k^2)\,, \tag{18}$$

and therefore

$$A = \frac{1}{n(n+2)} \int \frac{d^n k}{(2\pi)^n} k^4 f(k^2)\,, \tag{19}$$

as required.
Now let's consider $I_{\alpha\beta\gamma\delta\epsilon}$. Under a change of variables corresponding to a reflection, i.e. substituting $\tilde{k}_\alpha = -k_\alpha$, we have

$$k_\alpha k_\beta k_\gamma k_\delta k_\epsilon = -\tilde{k}_\alpha \tilde{k}_\beta \tilde{k}_\gamma \tilde{k}_\delta \tilde{k}_\epsilon\,, \tag{20}$$

$k^2 = \tilde{k}^2$, and

$$d^n k = \left|\frac{\partial \vec{k}}{\partial \vec{\tilde{k}}}\right| d^n \tilde{k} = |(-1)^n| d^n \tilde{k} = d^n \tilde{k}\,. \tag{21}$$

Thus, under the reflection change of variables,

$$I_{\alpha\beta\gamma\delta\epsilon} = \int \frac{d^n k}{(2\pi)^n} k_\alpha k_\beta k_\gamma k_\delta k_\epsilon f(k^2) = \int \frac{d^n \tilde{k}}{(2\pi)^n} (-\tilde{k}_\alpha \tilde{k}_\beta \tilde{k}_\gamma \tilde{k}_\delta \tilde{k}_\epsilon) f(\tilde{k}^2)\,. \tag{22}$$

Dropping the tilde in the last expression since it is simply an integration variable, we can see that $I_{\alpha\beta\gamma\delta\epsilon} = -I_{\alpha\beta\gamma\delta\epsilon} = 0$. More generally, this is true for any odd number of k_α's.

3 Moment-of-Inertia Tensor (20 points)

In this problem, you will construct the moment-of-inertia tensor for a 3D rigid body, with mass density given by $\rho(\vec{x})$. A rigid body can be thought of as a collection of particles with positions $\vec{x}_I$ (here I indexes the particles) that do not move relative to each other as the whole body moves. The length between any two particles is a constant, and the angles between the vectors connecting any two pairs of particles are also constant. This can be summarized by the condition that $(\vec{x}_I - \vec{x}_J) \cdot (\vec{x}_K - \vec{x}_L)$ remains constant for any four particles I, J, K, L for any moving rigid body.

Consider a rigid body moving in such a way that the position of the particles after some infinitesimal time dt is given by some linear transformation $T^i{}_j$, i.e. $x^i_I \mapsto T^i{}_j x^j_I$.

(a) Argue that T has to be a special orthogonal transformation, i.e. the matrix representing T is in $SO(3)$. This just means that rigid body is rotating about some arbitrary origin (the other possibility is translation, which is *not* a linear transformation, but we'll ignore it here).

SOLUTION: Under the transformation T, we see that

$$(\vec{x}_I - \vec{x}_J) \cdot (\vec{x}_K - \vec{x}_L) \mapsto (T(\vec{x}_I) - T(\vec{x}_J)) \cdot (T(\vec{x}_K) - T(\vec{x}_L)) \\ = T(\vec{x}_I - \vec{x}_J) \cdot T(\vec{x}_K - \vec{x}_L)\,, \tag{23}$$

In coordinates, this is

$$T^i{}_j\,(\vec{x}_I - \vec{x}_J)^j T_{ik}(\vec{x}_K - \vec{x}_L)^k = \delta_{jk}(\vec{x}_I - \vec{x}_J)^j(\vec{x}_K - \vec{x}_L)^k \tag{24}$$

for all I, J, K, L, from which we see that

$$T^i{}_j\,T_{ik} = \delta^i_k\,. \tag{25}$$

This means that the matrix T is orthogonal, i.e. $T^\intercal = T^{-1}$. It is easy to see that as $\mathrm{d}t \to 0$, T must tend to the identity, which has determinant 1, and therefore we need $\det(T) = 1$ as well. Thus, $T \in SO(3)$ as required.

(b) As $\mathrm{d}t \to 0$, T must tend toward the identity transformation. We can therefore write $T^i{}_j = \delta^i_j - A^i{}_j \mathrm{d}t$, for some components $A^i{}_j$. Use the fact that $T^i{}_j$ is orthogonal to show that $A^i{}_j$ is antisymmetric.

SOLUTION: From the orthogonality condition, we have

$$T^i{}_j\,T_{ik} = (\delta^i_j - A^i{}_j \mathrm{d}t)(\delta_{ik} - A_{ik}\mathrm{d}t) = \delta - (A_{jk} + A_{kj})\mathrm{d}t + O(\mathrm{d}t^2)\,. \tag{26}$$

Since this must equal δ_{jk} for all $\mathrm{d}t$, we see that $A_{jk} + A_{kj} = 0$, i.e. $A^i{}_j$ is antisymmetric as required.

(c) Show that the instantaneous velocity of particle I, v^i_I, can be written as $v^i_I = \epsilon^{ijk}\omega_j(x_I)_k$, where

$$\omega_i = \frac{1}{2}\epsilon_{ijk}A^{jk}\,, \tag{27}$$

for *any* particle I. Explain the physical meaning of the vector ω^i.

SOLUTION: Under the infinitesimal transformation T, we have

$$x^i_I \mapsto (T^i{}_j - A^i{}_j \mathrm{d}t)x^j_I = x^i_I - A^i{}_j x^j_I \mathrm{d}t\,, \tag{28}$$

and so the instantaneous velocity is given by

$$v^i_I = -A^i{}_j (x_I)^j\,. \tag{29}$$

Now,

$$\epsilon^{mnk}\omega_k = \frac{1}{2}\epsilon^{mnk}\epsilon_{kij}A^{ij} = \frac{1}{2}(\delta^m_i\delta^n_j - \delta^m_j\delta^n_i)A^{ij} = A^{mn}\,, \tag{30}$$

and so

$$v^i_I = -A^{ij}(x_I)_j = -\epsilon^{ijk}\omega_k(x_I)_j = \epsilon^{ijk}\omega_j(x_I)_k\,, \tag{31}$$

as required.
We see that in terms of vectors, $\vec{v}_I = \vec{\omega} \times \vec{x}_I$, and so $\vec{\omega}$ is the angular velocity vector of the rigid body.

(d) The angular momentum of a single particle is given by $L_I^i = \epsilon^{ijk}(x_I)_j(p_I)_k$, where $p_I^i = m_I v_I^i$ is the linear momentum of particle I. Let's consider the continuum limit, where we can break up the rigid body into infinitesimal volume elements with mass $\mathrm{d}^3\vec{x}\,\rho(\vec{x})$, and position $\vec{x}$. Show that the angular momentum of the purely rotating rigid body can be written as

$$L^i = \int \mathrm{d}^3\vec{x}\,\rho(\vec{x})\epsilon^{ijk}x_j v_k\,. \tag{32}$$

Hence, show that $L^i = I^{ij}\omega_j$, where

$$I^{ij} = \int \mathrm{d}^3\vec{x}\,\rho(\vec{x})(x^k x_k \delta^{ij} - x^i x^j) \tag{33}$$

is the moment-of-inertia tensor of the rigid body.

SOLUTION: For each volume element $\mathrm{d}^3\vec{x}$, the linear momentum is $\mathrm{d}^3\vec{x}\rho(\vec{x})v^i$, and the angular momentum is

$$dL^i = \epsilon^{ijk}x_j(\rho(\vec{x})\mathrm{d}^3\vec{x}\,v_k) \tag{34}$$

and hence integrating over all volume, we find

$$L^i = \int \mathrm{d}^3\vec{x}\,\rho(\vec{x})\epsilon^{ijk}x_j v_k\,. \tag{35}$$

However, we showed that $v_i = \epsilon^{ijk}\omega_j x_k$, and so

$$\begin{aligned} L^i &= \int \mathrm{d}^3\vec{x}\,\rho(\vec{x})\epsilon^{ijk}x_j\epsilon^{klm}\omega_l x_m \\ &= \int d^3\vec{x}\,\rho(\vec{x})\left(\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}\right)x_j\omega_l x_m \\ &= \int d^3\vec{x}\,\rho(\vec{x})(x^j x_j\omega^i - x^i x_j\omega^j) \\ &= \left(\int d^3\vec{x}\,\rho(\vec{x})(x^k x_k\delta^{ij} - x^i x^j)\right)\omega_j \\ &= I^{ij}\omega_j\,, \end{aligned} \tag{36}$$

where

$$I^{ij} = \int \mathrm{d}^3\vec{x}\,\rho(\vec{x})(x^k x_k\delta^{ij} - x^i x^j)\,, \tag{37}$$

as required.

(e) Find the moment-of-inertia tensor of a uniform density sphere of radius R and mass M. (*Hint:* isotropic tensors are your friends).

SOLUTION: With spherical symmetry, I expect the moment of inertia tensor to be an isotropic tensor, i.e. $I^i{}_j = C\delta^i_j$ for some constant C. To find C, let's take the trace of both sides.

$$\delta_{ji}I^{ij} = C\delta_{ji}\delta^{ij} = C\delta^j_j = 3C\,. \tag{38}$$

On the other hand, since the density is a constant ρ,

$$\begin{aligned} \delta_{ji} I^{ij} &= \int \mathrm{d}^3\vec{x}\, \rho (x^k x_k \delta_{ji}\delta^{ij} - \delta_{ji} x^i x^j) \\ &= \int \mathrm{d}^3\vec{x}\, \rho (3x^k x_k - x^j x_j) \\ &= 2\int \mathrm{d}^3\vec{x}\, \rho x^k x_k \,. \end{aligned} \tag{39}$$

Thus,

$$\begin{aligned} I^{ij} &= \frac{2}{3}\delta^{ij} \int \mathrm{d}^3\vec{x}\, \rho x^k x_k \\ &= \frac{2}{3}\delta^{ij} 4\pi\rho \int_0^R \mathrm{d}r\, r^4 \\ &= \frac{8\pi}{3}\rho \frac{R^5}{5}\delta^{ij} \,. \end{aligned} \tag{40}$$

However, since $\rho = M/(4\pi R^3/3)$, we have finally

$$I^{ij} = \frac{2}{5} M R^2 \delta^{ij} \,. \tag{41}$$

4 The Mechanics of Eigenvalues and Eigenvectors (15 points)

(a) Without using computer algebra system (it's fine to use it to check your answer), find the eigenvalues and corresponding eigenvectors of the following matrix:

$$M = \begin{pmatrix} 1 & -2 & -3 \\ 0 & 3 & 0 \\ 0 & -1 & -1 \end{pmatrix} \,. \tag{42}$$

SOLUTION:
The eigenvalues solve the following characteristic equation:

$$(1-\lambda)(3-\lambda)(-1-\lambda) = 0 \,, \tag{43}$$

and so the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = -1$ respectively. For each eigenvalue, we can find the corresponding eigenvector by solving $(M - \lambda_i I)\vec{x}_i = 0$. For λ_1, we have

$$\begin{pmatrix} 0 & -2 & -3 \\ 0 & 2 & 0 \\ 0 & -1 & -2 \end{pmatrix} \vec{x}_1 = 0 \,, \tag{44}$$

The second row tells us that $x_2 = 0$, while the first row tells us that $-2x_2 - 3x_3 = 0$, implying that $x_3 = 0$. Finally, the last row implies x_1 can have any value, and so we can set it to 1. Next, for $\lambda_2 = 3$, we have

$$\begin{pmatrix} -2 & -2 & -3 \\ 0 & 0 & 0 \\ 0 & -1 & -4 \end{pmatrix} \vec{x}_2 = 0 \,, \tag{45}$$

The third row says that $-x_2 - 4x_3 = 0$, while the first row gives $-2x_1 - 2x_2 - 3x_3 = -2x_1 + 5x_3 = 0$. Setting $x_3 = 2$ arbitrarily (eigenvectors are only defined up to overall normalization), we find

$x_1 = 5$, and $x_2 = -8$. Finally, for $\lambda_3 = -1$, we have

$$\begin{pmatrix} 2 & -2 & -3 \\ 0 & 4 & 0 \\ 0 & -1 & 0 \end{pmatrix} \vec{x}_3 = 0\,, \tag{46}$$

and so the second and third rows tell us that $x_2 = 0$, and now the first row says $2x_1 - 2x_2 - 3x_3 = 2x_1 - 3x_3 = 0$. We can thus choose $x_3 = 2$, and $x_1 = 3$. We therefore have the following eigenvalue/eigenvector pairs:

$$\begin{aligned} \lambda_1 &= 1\,, \quad \vec{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\,, \\ \lambda_2 &= 3\,, \quad \vec{x}_2 = \begin{pmatrix} 5 \\ -8 \\ 2 \end{pmatrix}\,, \\ \lambda_3 &= -1\,, \quad \vec{x}_3 = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}\,. \end{aligned} \tag{47}$$

(b) Consider a diagonalizable $n \times n$ matrix A, with eigenvalues λ_i, with $i = 1, \cdots, n$. Show that $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$, and that $\mathrm{Tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

SOLUTION:
Since A is diagonalizable, we can write $A = PDP^{-1}$, where D is a diagonal matrix with the eigenvalues λ_i on the diagonal. The determinant of A is then given by

$$\det(A) = \det(PDP^{-1}) = \det(P)\det(D)\det(P^{-1}) = \det(D) = \lambda_1 \lambda_2 \cdots \lambda_n\,, \tag{48}$$

where we have used the fact that $\det(P^{-1}) = 1/\det(P)$, and that the determinant of a diagonal matrix is simply the product of its diagonal elements.
Similarly, the trace of A is given by (this time in index notation)

$$\mathrm{Tr}(A) = A^i{}_i = P^i{}_j D^{jk} P^{-1}_{ki} = P^{-1}_{ki} P^i{}_j D^{jk} = \delta_{kj} D^{jk} = D^j{}_j = \lambda_1 + \lambda_2 + \cdots + \lambda_n\,. \tag{49}$$

(c) Suppose $\vec{v}$ is an eigenvector for the $n \times n$ matrices A and B. Is $\vec{v}$ an eigenvector for $A + B$? How about AB?

SOLUTION:
Suppose $A\vec{v} = \lambda \vec{v}$ and $B\vec{v} = \mu \vec{v}$. Then,

$$(A + B)\vec{v} = A\vec{v} + B\vec{v} = \lambda \vec{v} + \mu \vec{v} = (\lambda + \mu)\vec{v}\,, \tag{50}$$

and so $\vec{v}$ is indeed an eigenvector of $A + B$, with eigenvalue $\lambda + \mu$.
However, for the product, we have

$$AB\vec{v} = A(B\vec{v}) = A(\mu\vec{v}) = \mu(A\vec{v}) = \mu(\lambda\vec{v}) = (\mu\lambda)\vec{v}\,, \tag{51}$$

and so $\vec{v}$ is also an eigenvector of AB, with eigenvalue $\mu\lambda$.

(d) Suppose λ, μ are eigenvalues for the $n \times n$ matrix A. Is $\lambda + \mu$ an eigenvalue for A?

SOLUTION:
Not necessarily. For example, consider the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} , \tag{52}$$

which has eigenvalues $\lambda = 1$ and $\mu = 2$. However, $\lambda + \mu = 3$ is not an eigenvalue of A.

(e) Given that λ is an eigenvalue of A, find an eigenvalue of A^2.

SOLUTION:
If $A\vec{v} = \lambda\vec{v}$, then

$$A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v} \,, \tag{53}$$

and so λ^2 is an eigenvalue of A^2.

(f) Prove that $\det(A) = \det(A^\intercal)$, where $^\intercal$ is the transpose. Hence, show that A and $A^\intercal$ have the same eigenvalues.

SOLUTION:
We can write the determinant of ${(A^\intercal)^i}_j = {A_j}^i$ as

$$\begin{aligned} \det(A^\intercal) &= \frac{1}{n!}\epsilon^{i_1\cdots i_n}\epsilon_{j_1\cdots j_n}{(A^\intercal)^{j_1}}_{i_1}\cdots{(A^\intercal)^{j_n}}_{i_n} \\ &= \frac{1}{n!}\epsilon^{i_1\cdots i_n}\epsilon_{j_1\cdots j_n}{A_{i_1}}^{j_1}\cdots{A_{i_n}}^{j_n} \\ &= \frac{1}{n!}\epsilon_{i_1\cdots i_n}\epsilon^{j_1\cdots j_n}{A^{i_1}}_{j_1}\cdots{A^{i_n}}_{j_n} \\ &= \det(A) \,. \end{aligned} \tag{54}$$

The characteristic equation for $A^\intercal$ is

$$\det(A^\intercal - \lambda I) = 0 \implies \det((A - \lambda I)^\intercal) = 0 \implies \det(A - \lambda I) = 0 \,. \tag{55}$$

Here, we have used the fact that $(A+B)^\intercal = A^\intercal + B^\intercal$, and that the identity matrix is symmetric. We have shown that the characteristic equation for $A^\intercal$ is the same as that for A, and hence they have the same eigenvalues.

5 (SG 10.11) Invariant Content of a (1,1)-Tensor (10 points)

Let ${T^i}_j$ be the 3×3 array of components of a tensor. Consider the objects

$$a = {T^i}_i \,, \qquad b = {T^i}_j{T^j}_i \,, \qquad c = {T^i}_j{T^j}_k{T^k}_i \,. \tag{56}$$

Show explicitly that c is an invariant, i.e. show how each T transforms under a change of basis, and check that the transformation law reduces to that of an invariant. Describe these objects in terms of properties of the matrix ${T^i}_j$.

Assume that ${T^i}_j$ has 3 distinct eigenvalues. Show that the eigenvalues of the linear map represented by T can be found by solving the equation

$$\lambda^3 - a\lambda^2 + \frac{1}{2}(a^2 - b)\lambda - \frac{1}{6}(a^3 - 3ab + 2c) = 0 \,. \tag{57}$$

(*Hint:* Choose a good basis!)

SOLUTION:
Under a change of basis,

$$\begin{aligned} T^i{}_j T^j{}_k T^k{}_i &\mapsto a^{i'}{}_i (a^{-1})^j{}_{j'} T^i{}_j a^{j'}{}_l (a^{-1})^m{}_{k'} T^l{}_m a^{k'}{}_n (a^{-1})^p{}_{i'} T^n{}_p \\ &= a^{i'}{}_i (a^{-1})^j{}_{j'} a^{j'}{}_l (a^{-1})^m{}_{k'} a^{k'}{}_n (a^{-1})^p{}_{i'} T^i{}_j T^l{}_m T^n{}_p \\ &= \delta^j_l \delta^m_n \delta^p_i T^i{}_j T^l{}_m T^n{}_p \\ &= T^i{}_j T^j{}_k T^k{}_i \,, \end{aligned} \tag{58}$$

as required. In terms of the matrix $T^i{}_j$, we have $a = \mathrm{Tr}(T)$, $b = \mathrm{Tr}(T^2)$, and $c = \mathrm{Tr}(T^3)$.
The eigenvalues of the linear map represented by T are the roots of the characteristic equation

$$\det(T - \lambda I) = 0 \tag{59}$$

for T in any basis. Since the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are distinct, T is diagonalizable, and hence we can choose the eigenbasis where $T^i{}_j = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. Therefore, the eigenvalues satisfy the equation

$$(\lambda - \tau_1)(\lambda - \tau_2)(\lambda - \tau_3) = 0 \tag{60}$$

where I have simplified the notation so that $\tau_i = T^i{}_i$, with *no summation being implied here.* Also in this basis,

$$a = \tau_1 + \tau_2 + \tau_3 \,, \quad b = \tau_1^2 + \tau_2^2 + \tau_3^2 \,, \quad c = \tau_1^3 + \tau_2^3 + \tau_3^3 \,. \tag{61}$$

From these relations, we can see that

$$\tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1 = \frac{1}{2}\left[(\tau_1 + \tau_2 + \tau_3)^2 - (\tau_1^2 + \tau_2^2 + \tau_3^2)\right] = \frac{1}{2}(a^2 - b) \,, \tag{62}$$

and

$$\begin{aligned} 6\tau_1\tau_2\tau_3 = (\tau_1 + \tau_2 + \tau_3)^3 - (\tau_1^3 + \tau_2^3 + \tau_3^3) \\ - 3(\tau_1 + \tau_2 + \tau_3)(\tau_1^2 + \tau_2^2 + \tau_3^2) + 3(\tau_1^3 + \tau_2^3 + \tau_3^3) \\ = a^3 + 2c - 3ab \end{aligned} \tag{63}$$

Thus, we finally have

$$(\lambda - \tau_1)(\lambda - \tau_2)(\lambda - \tau_3) = \lambda^3 - \lambda^2 a + \frac{1}{2}(a^2 - b)\lambda - \frac{1}{6}(a^3 - 3ab + 2c) = 0 \,. \tag{64}$$

as required.