Midterm 1 (Total Time: 1 h 40 min)

Question	**Max. Points**
Q1	15
Q2	20
Q3	20
Q4	30
Q5	30
Total	**115**

1. Do not turn the page until you are instructed to do so.
2. Fill in your name **with your student ID**, and for subject please indicate "PY501 Fall 2025 Midterm 1". All other fields can be left blank.
3. **All of your work must be written in the blue exam book.** This document only contains the questions, and will **not** be collected from you.
4. Hand your blue exam booklet in at the end of the exam and before you leave the room.
5. This exam contains **7** pages (including this cover page) and **5** problems.
6. This is a closed book exam. **No books, notes or calculators allowed**.
7. **Some useful facts are provided to you on the page behind this**. Not all of them may be useful, but you may use them as you see fit.
8. Answers without supporting work will receive no credit. **Conversely, supporting work will be given plenty of partial credit, even if the final answer is incorrect.**
9. You will be graded based solely on what the grader can understand of your solutions.
10. Points allocated to each question are indicated in brackets. **Please plan your time appropriately.**

Useful Information

1. Three-dimensional Levi-Civita symbol identities:

$$\begin{aligned}
\epsilon_{ijk}\epsilon_{lmn} &= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \\
\epsilon^{ijk}\epsilon_{pqk} &= \delta^i_p\delta^j_q - \delta^i_q\delta^j_p \\
\epsilon_{jmn}\epsilon^{imn} &= 2\delta^i_j \\
\epsilon^{ijk}\epsilon_{ijk} &= 6
\end{aligned}$$

2. The most general $O(n)$ invariant, rank-4 Cartesian tensor T_{ijkl} can be written as

$$T_{ijkl} = a\delta_{ij}\delta_{kl} + b\delta_{ik}\delta_{jl} + c\delta_{il}\delta_{jk}$$

where a, b, and c are constants.

3. The determinant of an $n \times n$ matrix M can be written as

$$\epsilon_{\mu_1\cdots\mu_n}\det(M) = \epsilon_{\nu_1\cdots\nu_n}M^{\nu_1}{}_{\mu_1}\cdots M^{\nu_n}{}_{\mu_n}\,, \tag{1}$$

or equivalently

$$\det(M) = \frac{1}{n!}\epsilon^{\mu_1\cdots\mu_n}\epsilon_{\nu_1\cdots\nu_n}M^{\nu_1}{}_{\mu_1}\cdots M^{\nu_n}{}_{\mu_n}\,. \tag{2}$$

4. An $n \times n$ Hermitian complex matrix is diagonalizable into an orthonormal eigenbasis, with n real eigenvalues. The same is true for a real symmetric matrix.

5. Given a functional $S[q] = \int dt\, L(t; q, \dot{q})$, the functional is extremized for fixed endpoint variations of $q(t)$ satisfying the Euler-Lagrange equation,

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0\,. \tag{3}$$

6. Consider a Lagrangian $L(t; q^i, \dot{q}^i)$ where q^i are the generalized coordinates. Given that the transformation of all of the coordinates $q^i \mapsto q^i + \varepsilon\eta^i$ is a symmetry of the action, with ϵ being an infinitesimal quantity, we can define the Noether charge

$$Q = K - \frac{\partial L}{\partial \dot{q}^i}\eta^i\,,$$

where the action transforms as

$$\delta S = \varepsilon\int dt\,\frac{dK}{dt}\,.$$

A similar result holds for continuous systems.

7. The first integral of the system defined by the functional

$$J[y] = \int dx\, f(x; y, y')$$

is

$$I \equiv y'\frac{\partial f}{\partial y'} - f\,.$$

1 Maxwell's Equations (15 points)

Maxwell's equations in vacuum written in vector notation is given by

$$\nabla \cdot \vec{E} = 0\,, \tag{4}$$
$$\nabla \cdot \vec{B} = 0\,, \tag{5}$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\,, \tag{6}$$
$$\nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t}\,. \tag{7}$$

(a) *(5 points)* Write these equations in index notation, using the Levi-Civita symbol ϵ_{ijk}.

SOLUTION:

$$\partial_i E^i = 0\,, \tag{8}$$
$$\partial_i B^i = 0\,, \tag{9}$$
$$\epsilon_{ijk}\partial^j E^k = -\partial_t B_i\,, \tag{10}$$
$$\epsilon_{ijk}\partial^j B^k = \partial_t E_i\,. \tag{11}$$

(b) *(10 points)* By taking the curl (i.e. $\nabla \times \cdot$) of Eq. (6), and using relevant identities of the Levi-Civita symbol, show that the electric field $\vec{E}$ satisfies the wave equation,

$$\nabla^2 \vec{E} - \frac{\partial^2 \vec{E}}{\partial t^2} = 0\,. \tag{12}$$

SOLUTION:
Taking the curl of Eq. (6), we have

$$\begin{aligned}
\epsilon^{mni}\partial_n(\epsilon_{ijk}\partial^j E^k) &= -\epsilon^{mni}\partial_n\partial_t B_i \\
\left(\delta^m_j \delta^n_k - \delta^m_k \delta^n_j\right)\left(\partial_n \partial^j E^k\right) &= -\partial_t \epsilon^{mni}\partial_n B_i \\
\partial^m \partial_k E^k - \partial^k \partial_k E^m &= -\partial_t \epsilon^{mni}\partial_n B_i\,.
\end{aligned} \tag{13}$$

From Gauss' Law for electric fields, Eq. (4), the first term on the left-hand side vanishes. Using Ampere's Law, we can write the right-hand side as $-\partial_t(\partial_t E^m)$, and so we have

$$-\partial^k \partial_k E^m = -\partial_t^2 E^m\,, \tag{14}$$

or in vector notation,

$$\nabla^2 \vec{E} - \frac{\partial^2 \vec{E}}{\partial t^2} = 0\,. \tag{15}$$

as required.

2 An Assortment of Tensors (20 points)

(a) *(5 points)* The tensor with components T_{ijkl} is completely antisymmetric with respect to all pairs of indices in a 3-dimensional vector space V. Determine the number of nonzero independent components T_{ijkl} has, and explain your answer.

SOLUTION: By the pigeonhole principle, at least one the indices i, j, k, l is repeated. Swapping those two indices picks up a minus sign, but leaves the tensor unchanged. Therefore, $T_{ijkl} = 0$, and there are no nonzero independent components.

(b) *(10 points)* The stress-energy tensor of a perfect fluid in Minkowski space, with metric

$$\eta_{\mu\nu} \equiv \text{diag}(+1, -1, -1, -1)\,, \tag{16}$$

is given by

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p\eta^{\mu\nu}\,, \tag{17}$$

where ρ is the energy density, and p is the pressure. u^μ is a four component vector known as the four-velocity of the fluid, with $u^0 = \gamma$, and $u^i = \gamma v^i$ for $i = 1, 2, 3$. $\gamma \equiv 1/\sqrt{1 - |\vec{v}|^2}$, where $\vec{v}$ is the 3D velocity of the fluid. Evaluate the trace $T^\mu{}_\mu$.

SOLUTION: We can evaluate the trace in any coordinate basis, so let's do it in the rest frame of the fluid element. This means that $u^\mu = (1, 0, 0, 0)$. The trace is then

$$T^\mu{}_\mu = (\rho + p)u^\mu u_\mu - p\eta^{\mu\nu}\eta_{\mu\nu} = \rho + p - p\delta^\mu_\mu = \rho - 3p\,. \tag{18}$$

(c) *(5 points)* Prove the cyclic property of traces: for any set of $n \times n$ matrices $M_1, \cdots, M_n$,

$$\text{Tr}(M_1 M_2 \cdots M_n) = \text{Tr}(M_n M_1 \cdots M_{n-1})\,, \tag{19}$$

with the matrices being multiplied together in the usual way.

SOLUTION: The product can be written as

$$\begin{aligned}\text{Tr}(M_1 M_2 \cdots M_n) &= {M_1}^{\mu_1}{}_{\mu_2} {M_2}^{\mu_2}{}_{\mu_3} \cdots {M_n}^{\mu_n}{}_{\mu_1} \\ &= {M_n}^{\mu_n}{}_{\mu_1} {M_1}^{\mu_1}{}_{\mu_2} {M_2}^{\mu_2}{}_{\mu_3} \cdots {M_{n-1}}^{\mu_{n-1}}{}_{\mu_n} \\ &= \text{Tr}(M_n M_1 \cdots M_{n-1})\,. \end{aligned} \tag{20}$$

3 Eigenvalues and Eigenvectors (20 points)

(a) *(10 points)* Consider the matrix

$$M = \begin{pmatrix} 2 & -1 \\ -4 & 5 \end{pmatrix} . \tag{21}$$

Find the eigenvalues and corresponding eigenvectors of M.

SOLUTION:
The eigenvalues are the roots of the characteristic polynomial,

$$\begin{aligned} \det(M - \lambda I) &= \begin{pmatrix} 2-\lambda & -1 \\ -4 & 5-\lambda \end{pmatrix} = (2-\lambda)(5-\lambda) - (-1)(-4) \\ &= (2-\lambda)(5-\lambda) - 4 \\ &= (10 - 2\lambda - 5\lambda + \lambda^2) - 4 \\ &= \lambda^2 - 7\lambda + 6 \end{aligned} \tag{22}$$

Setting this equal to zero,

$$\lambda^2 - 7\lambda + 6 = 0 \implies (\lambda - 6)(\lambda - 1) = 0 \tag{23}$$

So the eigenvalues are 6 and 1.
For $\lambda = 6$:

$$(M - 6I)\vec{v} = 0 \implies \begin{pmatrix} -4 & -1 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \tag{24}$$

This gives $-4v_1 - v_2 = 0 \implies v_2 = -4v_1$. So an eigenvector is any multiple of $(1, -4)^\intercal$.
For $\lambda = 1$:

$$(M - I)\vec{v} = 0 \implies \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \tag{25}$$

This gives $v_1 - v_2 = 0 \implies v_1 = v_2$. So an eigenvector is any multiple of $(1, 1)^\intercal$.

(b) *(10 points)* Let A and B be two $n \times n$ Hermitian matrices. Suppose $AB = BA$, i.e. A and B commute. Show that A and B can be simultaneously diagonalized, i.e. they can be written as $A = PD_AP^{-1}$ and $B = PD_BP^{-1}$, where P is an invertible matrix, and D_A and D_B are diagonal matrices.

SOLUTION:
Let $\vec{x}$ be an eigenvector of B with eigenvalue λ, i.e. $B\vec{x} = \lambda\vec{x}$. Then

$$B(A\vec{x}) = A(B\vec{x}) \implies B(A\vec{x}) = \lambda(A\vec{x}) . \tag{26}$$

This shows that $A\vec{x}$ is an eigenvector of B with the same eigenvalue λ. But B as a Hermitian matrix has n unique eigenvalues, which means that $A\vec{x}$ must be proportional to $\vec{x}$, i.e. $A\vec{x} = \mu\vec{x}$ for some scalar μ. Therefore, $\vec{x}$ is also an eigenvector of A. Since there are n unique eigenvalues of B, there are n linearly independent eigenvectors $\vec{x}_1, \cdots, \vec{x}_n$ which are orthogonal, and B is diagonalizable into PD_BP^{-1}, where $P = (\vec{x}_1\, \vec{x}_2\, \cdots\, \vec{x}_n)$. But each eigenvector of B is also an eigenvector of A, and hence we can also write A as PD_AP^{-1}, where D_A is diagonal.

4 The Determinant (30 points)

Consider a rank-2 tensor in Euclidean $\mathbb{R}^3$, B_{ij}, with metric tensor δ_{ij}. The determinant is therefore given as

$$\det(B) = \frac{1}{3!}\epsilon^{ijk}\epsilon^{lmn}B_{li}B_{mj}B_{nk}\,, \tag{27}$$

or equivalently

$$\epsilon_{lmn}\det(B) = \epsilon^{ijk}B_{li}B_{mj}B_{nk}\,. \tag{28}$$

(a) *(15 points)* Show that the determinant is invariant under a change of basis corresponding to a rotation O, which are special orthogonal matrices with $O^\intercal = O^{-1}$, and $\det(O) = 1$ (i.e. SO(3) matrices).

SOLUTION:
Under a change of basis, we have

$$\begin{aligned}\epsilon^{ijk}\epsilon^{lmn}B_{li}B_{mj}B_{nk} &\mapsto \epsilon^{ijk}\epsilon^{lmn}B'_{li}B'_{mj}B'_{nk} \\ &= \epsilon^{ijk}\epsilon^{lmn}O^{l'}{}_{l}O^{i'}{}_{i}B_{l'i'}O^{m'}{}_{m}O^{j'}{}_{j}B_{m'j'}O^{n'}{}_{n}O^{k'}{}_{k}B_{n'k'}\,.\end{aligned} \tag{29}$$

However, we note that

$$\epsilon^{ijk}O^{i'}{}_{i}O^{j'}{}_{j}O^{k'}{}_{k} = \det(O)\epsilon^{i'j'k'} = \epsilon^{i'j'k'}\,, \tag{30}$$

since $\det(O) = 1$. Therefore, the expression above simplifies to

$$\begin{aligned}\epsilon^{ijk}\epsilon^{lmn}B_{li}B_{mj}B_{nk} &\mapsto \epsilon^{ijk}\epsilon^{lmn}B'_{li}B'_{mj}B'_{nk} \\ &= \epsilon^{i'j'k'}\epsilon^{lmn}O^{l'}{}_{l}B_{l'i'}O^{m'}{}_{m}B_{m'j'}O^{n'}{}_{n}B_{n'k'}\,.\end{aligned} \tag{31}$$

Similarly,

$$\epsilon^{lmn}O^{l'}{}_{l}O^{m'}{}_{m}O^{n'}{}_{n} = \det(O)\epsilon^{l'm'n'} = \epsilon^{l'm'n'}\,. \tag{32}$$

Therefore,

$$\begin{aligned}\epsilon^{ijk}\epsilon^{lmn}B_{li}B_{mj}B_{nk} &\mapsto \epsilon^{i'j'k'}\epsilon^{l'm'n'}B_{l'i'}B_{m'j'}B_{n'k'} \\ &= \epsilon^{ijk}\epsilon^{lmn}B_{li}B_{mj}B_{nk}\,,\end{aligned} \tag{33}$$

thus showing that the determinant is invariant under rotations.

(b) *(15 points)* Show that for any two 3×3 matrices A and B, $\det(AB) = \det(A)\det(B)$, where AB corresponds to the usual matrix multiplication of A and B, i.e. $(AB)^i{}_j = A^{ik}B_{kj}$. Starting from Eq. (28) is probably a little easier.

SOLUTION:
Starting from Eq. (28), we have

$$\begin{aligned}\det(AB)\epsilon_{lmn} &= \epsilon^{ijk}(AB)_{li}(AB)_{mj}(AB)_{nk} \\ &= \epsilon^{ijk}A_l{}^{q}B_{qi}A_m{}^{r}B_{rj}A_n{}^{s}B_{sk} \\ &= \epsilon^{ijk}B_{qi}B_{rj}B_{sk}\cdot A_l{}^{q}A_m{}^{r}A_n{}^{s} \\ &= \det(B)\epsilon_{qrs}A_l{}^{q}A_m{}^{r}A_n{}^{s} \\ &= \det(B)\epsilon^{qrs}A_{lq}A_{mr}A_{ns} \\ &= \det(B)\det(A)\epsilon_{lmn}\,.\end{aligned} \tag{34}$$

Therefore, we find that

$$\det(AB) = \det(A)\det(B)\,, \tag{35}$$

as required.

5 Kepler Problem (30 points)

Consider two masses in 3D space, with masses m_1 and m_2, and positions $\vec{r}_1$ and $\vec{r}_2$, interacting with each other through a potential that depends only on the relative distance between them, i.e. $V \equiv V(|\vec{r}_1 - \vec{r}_2|)$. The Lagrangian for this system is

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2^2 - V(|\vec{r}_1 - \vec{r}_2|) \,, \tag{36}$$

with corresponding action

$$S = \int dt \, L \,. \tag{37}$$

(a) *(10 points)* Introduce the coordinates $\vec{R}$ and $\vec{r}$, defined as

$$\vec{R} \equiv \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \,, \tag{38}$$

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2 \,. \tag{39}$$

Show that the Lagrangian can be written as

$$L = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 - V(|\vec{r}|) \,, \tag{40}$$

where $M \equiv m_1 + m_2$, and $\mu \equiv m_1 m_2 / (m_1 + m_2)$. You can start from the final expression and work backwards if you find that easier.

SOLUTION:
First, let's write

$$\dot{\vec{R}} = \frac{m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2}{m_1 + m_2} = \frac{\mu}{m_2} \vec{r}_1 + \frac{\mu}{m_1} \vec{r}_2 \,. \tag{41}$$

Then

$$\begin{aligned} \frac{1}{2} M \dot{\vec{R}}^2 &= \frac{1}{2} (m_1 + m_2) \left(\frac{\mu^2}{m_2^2} \dot{\vec{r}}_1^2 + \frac{\mu^2}{m_1^2} \dot{\vec{r}}_2^2 + 2 \frac{\mu^2}{m_1 m_2} \dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2 \right) \\ &= \frac{1}{2} \frac{m_1}{m_2} \mu \dot{\vec{r}}_1^2 + \frac{1}{2} \frac{m_2}{m_1} \mu \dot{\vec{r}}_2^2 + \mu \dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2 \,. \end{aligned} \tag{42}$$

On the other hand,

$$\begin{aligned} \frac{1}{2} \mu \dot{\vec{r}}^2 &= \frac{1}{2} \mu (\dot{\vec{r}}_1 - \dot{\vec{r}}_2)^2 \\ &= \frac{1}{2} \mu (\dot{\vec{r}}_1^2 + \dot{\vec{r}}_2^2 - 2 \dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2) \,. \end{aligned} \tag{43}$$

And so,

$$\begin{aligned} \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 - V(|\vec{r}|) &= \frac{\mu}{2} \left(\frac{m_1}{m_2} + 1 \right) \dot{\vec{r}}_1^2 + \frac{\mu}{2} \left(\frac{m_2}{m_1} + 1 \right) \dot{\vec{r}}_2^2 + (\mu - \mu) \dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2 - V(|\vec{r}_1 - \vec{r}_2|) \\ &= \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2^2 - V(|\vec{r}_1 - \vec{r}_2|) \,. \end{aligned} \tag{44}$$

(b) *(10 points)* Obtain the equations of motion for $\vec{R}$ and $\vec{r}$ in terms of derivatives of V (i.e. $V'(|\vec{r}|)$, $V''(|\vec{r}|)$ etc.). Give the physical meaning of $\vec{R}$ and its equation of motion.

SOLUTION:
The Euler-Lagrange equations for $\vec{R}$ gives

$$\frac{d}{dt}(M\dot{\vec{R}}) = 0 \implies M\ddot{\vec{R}} = 0\,, \tag{45}$$

while the Euler-Lagrange equations for $\vec{r}$ gives

$$-V'(|\vec{r}|)\cdot\frac{\vec{r}}{|\vec{r}|} - \frac{d}{dt}(\mu\dot{\vec{r}}) = 0 \implies \mu\ddot{\vec{r}} = -V'(|\vec{r}|)\cdot\frac{\vec{r}}{|\vec{r}|}\,. \tag{46}$$

$\vec{R}$ is the center-of-mass coordinate, and the equation of motion states that the center of mass moves with constant velocity.

(c) *(10 points)* Find **three** distinct symmetries of the action, and determine their respective conserved quantities. You do **not** have to prove explicitly that the action is invariant, but you must explain why they are symmetries. Conserved quantities that correspond to components of a single vector only counts as one symmetry.

SOLUTION:
There is no explicit time dependence in the Lagrangian, and so the first integral i.e. the energy is conserved. The corresponding conserved quantity is

$$\frac{\partial L}{\partial\dot{R}^i}\dot{R}^i + \frac{\partial L}{\partial\dot{r}^i}\dot{r}^i - L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 + V(|\vec{r}|)\,. \tag{47}$$

$\vec{R}$ is also a cyclic coordinate, and so the momentum associated with it is conserved, i.e. the conserved charge is

$$\frac{\partial L}{\partial\dot{R}^i} = M\dot{R}^i\,. \tag{48}$$

This is identical to the statement that the center of mass moves with constant velocity, which we already figured out in the last part.
Finally, in spherical coordinates, we can write

$$\dot{\vec{r}}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\cdot\dot{\phi}^2\,, \tag{49}$$

Thus, ϕ is a cyclic coordinate. The conserved charge associated with ϕ is

$$\frac{\partial L}{\partial\dot{\phi}} = \mu r^2\sin^2\theta\cdot\dot{\phi}\,, \tag{50}$$

which is the angular momentum in the direction along $\vec{r}$.