Problem Set 9: Fourier Analysis

1 Contour Integration with Branch Cuts (20 points)

Show that

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}} \tag{1}$$

by choosing an appropriate branch cut, and evaluating the integral using:

(a) The contour shown on the left in Fig. 1.

SOLUTION:
First, we should note that for $z = re^{i\theta}$,

$$\begin{aligned} z^{1/2} = e^{\frac{1}{2}\log z} &= e^{\frac{1}{2}(\ln r + i(\theta+2n\pi))}\,,\ n \in \mathbb{Z} \\ &= r^{1/2}e^{i\theta/2}e^{in\pi}\,,\ n \in \mathbb{Z}\,. \end{aligned} \tag{2}$$

In order to avoid a multivalued function, we have to choose a branch cut for $\sqrt{z}$. For the first contour, we choose $-\pi/2 < \theta < 3\pi/2$, so that the branch cut lies along the negative imaginary axis, which allows for the contour to avoid the branch cut. Defining

$$g(z) = \frac{1}{\sqrt{z}(z^2+1)}\,, \tag{3}$$

Now, the only pole inside the contour as shown is at $z = i$, and so the integral along the contour is

$$\int_{-R}^{-\rho} \frac{dx}{\sqrt{x}(x^2+1)} + \int_{C_\rho} dz\, g(z) + \int_\rho^R \frac{dx}{\sqrt{x}(x^2+1)} + \int_{C_R} dz\, g(z) = 2\pi i \operatorname*{Res}_{z=i} g(z)\,, \tag{4}$$

with

$$\operatorname*{Res}_{z=i} g(z) = \frac{1}{\sqrt{i}(i+i)} = \frac{1}{2i}e^{-i\pi/4}\,. \tag{5}$$

Now, let's look at the contour integral over C_R. We have

$$\int_{C_R} dz\, g(z) = \int_0^\pi d\theta\, \frac{iRe^{i\theta}}{\sqrt{R}e^{i\theta/2}(R^2e^{2i\theta}+1)} = \int_0^\pi d\theta\, \frac{i\sqrt{R}e^{i\theta/2}}{R^2e^{2i\theta}+1}\,, \tag{6}$$

so that by the modulus inequality,

$$\left|\int_{C_R} dz\, g(z)\right| \leq \int_0^\pi d\theta \frac{R}{|R^2e^{2i\theta}+1|} \leq \int_0^\pi d\theta \frac{R}{\sqrt{R^4+2R^2\cos 2\theta+1}} \leq \int_0^\pi d\theta \frac{R}{R^2-1}\,, \tag{7}$$

so that

$$\lim_{R\to\infty}\left|\int_{C_R} dz\, g(z)\right| = 0 \tag{8}$$

At the same time, we can also plainly see that

$$\lim_{\rho\to 0}\int_{C_\rho} dz\, g(z) = \lim_{\rho\to 0}\int_\pi^0 d\theta \frac{i\sqrt{\rho}e^{i\theta/2}}{\rho^2e^{2i\theta}+1} = 0\,, \tag{9}$$

as the integrand simply vanishes as $\rho \to 0$. Therefore, as we take $\rho = 0$ and $R \to \infty$, we find

$$\lim_{\rho \to 0} \lim_{R \to \infty} \left(\int_{-R}^{-\rho} \frac{dx}{\sqrt{x}(x^2+1)} + \int_{\rho}^{R} \frac{dx}{\sqrt{x}(x^2+1)} \right) = \pi e^{-i\pi/4} \tag{10}$$

Notice that the first integral contains the square root of a negative number in the denominator. We can rewrite this as

$$\int_{-R}^{-\rho} \frac{dx}{\sqrt{x}(x^2+1)} = \int_{-R}^{-\rho} \frac{dx}{i\sqrt{-x}(x^2+1)} = -i \int_{\rho}^{R} \frac{dx}{\sqrt{x}(x^2+1)}\,, \tag{11}$$

where in the last line we made the change of variables $x \to -x$. Taking the limit, we therefore find

$$(1-i) \int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \pi e^{-i\pi/4} \implies \int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}\,, \tag{12}$$

as required.

(b) The contour shown on the right in Fig. 1.

SOLUTION:
For this contour, we now choose the branch cut $0 < \theta < 2\pi$. There are now two poles inside the contour, $z = \pm i$, and we find that

$$\operatorname*{Res}_{z=i} g(z) + \operatorname*{Res}_{z=-i} g(z) = \frac{1}{\sqrt{i}(i+i)} + \frac{1}{\sqrt{-i}(-i-i)} = \frac{1}{2i}\left(e^{-i\pi/4} - e^{-i3\pi/4}\right) = \frac{1}{\sqrt{2}i}\,. \tag{13}$$

First, let's consider the integral over C_R. This has the same result as before, except that the limits of the integration goes from 0 to 2π, but the conclusion remains the same, which is that

$$\lim_{R \to \infty} \int_{C_R} dz\, g(z) = 0\,. \tag{14}$$

Similarly, the contour integral over C_ρ has the same form as above, just that the limits of the integration are now 2π to 0.

$$\lim_{\rho \to 0} \int_{C_\rho} dz\, g(z) = 0\,. \tag{15}$$

That leaves the two straight contour integrals. The one in the positive direction, which lies just above the branch cut, simply has $\sqrt{x}$ leading to the positive square root. For the one in the negative direction, however, we have $\arg(z) = 2\pi$ along the contour, and so we should take the negative square root along that contour. Putting everything together, we find that as $\rho \to 0$ and $R \to \infty$, we have

$$\lim_{\rho \to 0} \lim_{R \to \infty} \left(\int_{\rho}^{R} \frac{dx}{\sqrt{x}(x^2+1)} + \int_{R}^{\rho} \frac{dx}{-\sqrt{x}(x^2+1)} \right) = 2\pi i \cdot \frac{1}{\sqrt{2}i} = \sqrt{2}\pi\,, \tag{16}$$

or in other words

$$2 \int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \sqrt{2}\pi \implies \int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}\,, \tag{17}$$

as required.

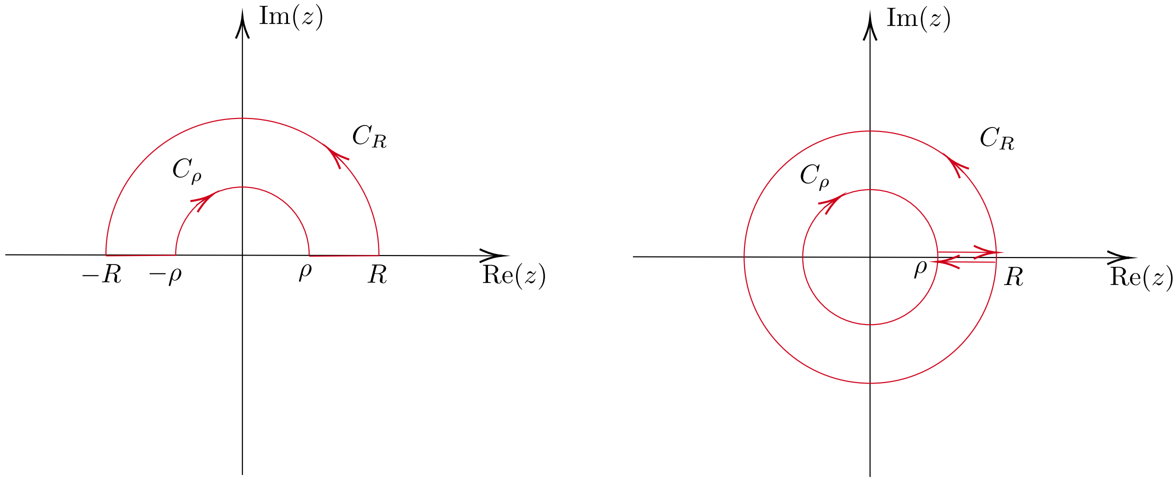

Figure 1: Contours for the problem set.

2 The Sawtooth Function (10 points)

Let $f(x)$ be the sawtooth function, defined by $f(x) = x$ for $-\pi < x < \pi$. Let $g(x) = \sin x$.

(a) Find $\tilde{f}_n$ the Fourier coefficients of $f(x)$, by expanding in the basis of $\exp(inx)/\sqrt{2\pi}$.

SOLUTION:
The Fourier coefficients are given by

$$\tilde{f}_n = \int_{-\pi}^{\pi} dx\, \frac{e^{inx}}{\sqrt{2\pi}} x\,. \tag{18}$$

We can see that $\tilde{f}_0 = 0$. For $n \neq 0$, we have

$$\begin{aligned} \tilde{f}_n &= \frac{x}{\sqrt{2\pi}} \frac{e^{inx}}{in}\bigg|_{-\pi}^{\pi} - \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} dx \frac{e^{inx}}{in} \\ &= \frac{x}{\sqrt{2\pi}} \frac{e^{inx}}{in}\bigg|_{-\pi}^{\pi} - \frac{1}{\sqrt{2\pi}} \left[-\frac{e^{inx}}{n^2} \right]_{-\pi}^{\pi} \\ &= \frac{\pi}{in\sqrt{2\pi}} (e^{in\pi} + e^{-in\pi}) \\ &= \frac{\sqrt{2\pi}}{in} \cos(n\pi) \\ &= -i\sqrt{2\pi} \frac{(-1)^n}{n}\,, n \neq 0\,. \end{aligned} \tag{19}$$

(b) Prove Parseval's identity for the Fourier series,

$$\int_{-\pi}^{\pi} dx\, |f(x)|^2 = \sum_{n=-\infty}^{\infty} |\tilde{f}_n|^2\,, \tag{20}$$

and use this to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,. \tag{21}$$

SOLUTION:

$$\begin{aligned}\int_{-\pi}^{\pi} dx\, |f(x)|^2 &= \int_{-\pi}^{\pi} dx\, \sum_m \tilde{f}_m^* \phi_m^*(x) \sum_n \tilde{f}_n \phi_n(x) \\ &= \sum_m \sum_n \tilde{f}_m^* \tilde{f}_n \int_{-\pi}^{\pi} dx\, \phi_m^*(x) \phi_n(x) \\ &= \sum_m \sum_n \tilde{f}_m^* \tilde{f}_n \delta_{mn} \\ &= \sum_m |\tilde{f}_m|^2 \,. \end{aligned} \tag{22}$$

By direct integration, we find

$$\int_{-\pi}^{\pi} dx\, |f(x)|^2 = \int_{-\pi}^{\pi} dx\, x^2 = \left. \frac{x^3}{3} \right|_{-\pi}^{\pi} = \frac{2\pi^3}{3} \,. \tag{23}$$

On the other hand, we have

$$\begin{aligned}\sum_{m=-\infty}^{\infty} |\tilde{f}_m|^2 &= \sum_{m \neq 0} \left| -i\sqrt{2\pi} \frac{(-1)^m}{m} \right|^2 \\ &= \sum_{m \neq 0} \frac{2\pi}{m^2} \\ &= 2 \sum_{m=1}^{\infty} \frac{2\pi}{m^2} \,. \end{aligned} \tag{24}$$

Therefore,

$$4\pi \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{2\pi^2}{3} \implies \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6} \,, \tag{25}$$

as required.

3 Poisson Summation Formula (15 points)

Let $f : \mathbb{R} \to \mathbb{C}$ be related to its Fourier transform via the usual inverse Fourier transform,

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k) \,. \tag{26}$$

(a) Show that the following result, known as the Poisson summation formula, holds:

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{n=-\infty}^{\infty} \tilde{f}(2\pi n) \,. \tag{27}$$

SOLUTION:

We see immediately that

$$\begin{aligned}\sum_{m=-\infty}^{\infty} f(m) &= \sum_m \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikm} \tilde{f}(k) \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{f}(k) \sum_m e^{ikm} \,. \end{aligned} \tag{28}$$

However, we know from lecture that

$$\frac{1}{L} \sum_{m=-\infty}^{\infty} e^{2\pi i m x/L} = \sum_{n=-\infty}^{\infty} \delta(x - nL) \,, \tag{29}$$

and so setting $L = 2\pi$, we have

$$\sum_{m=-\infty}^{\infty} e^{ikx} = 2\pi \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n) \,. \tag{30}$$

Thus, we see that

$$\sum_{m=-\infty}^{\infty} f(m) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{f}(k) \cdot 2\pi \sum_{n=-\infty}^{\infty} \delta(k - 2\pi n) = \sum_{n=-\infty}^{\infty} \tilde{f}(2\pi n) \,, \tag{31}$$

as required.

(b) Use the Poisson summation formula for the function $f(x) = \exp(-a|x|)$ for $a > 0$ to prove the following identity:

$$\sum_{n=-\infty}^{\infty} \frac{2a}{(2\pi n)^2 + a^2} = \coth(a/2) \,. \tag{32}$$

SOLUTION:
Let's begin by finding the Fourier transform of $\exp(-a|x|)$. We have

$$\begin{aligned}\int_{-\infty}^{\infty} dx\, e^{-ikx} e^{-a|x|} &= \int_{-\infty}^{0} dx\, e^{-ikx} e^{ax} + \int_{0}^{\infty} dx\, e^{-ikx} e^{-ax} \\ &= \int_0^{\infty} dx\, \left(e^{ikx} + e^{-ikx}\right) e^{-ax} \\ &= \left[\frac{e^{(ik-a)x}}{ik - a} + \frac{e^{-(ik+a)x}}{-(ik+a)} \right]_0^{\infty} \,. \end{aligned} \tag{33}$$

Now, we note that

$$\lim_{x\to\infty} \left| e^{(ik-a)x} \right| = \lim_{x\to\infty} e^{-ax} = 0 \,, \tag{34}$$

and therefore

$$\mathcal{F} e^{-a|x|} = \frac{1}{ik + a} - \frac{1}{ik - a} = \frac{2a}{a^2 + k^2} \,. \tag{35}$$

Therefore, by the Poisson summation formula, we have

$$\sum_{m=-\infty}^{\infty} e^{-a|m|} = \sum_{n=-\infty}^{\infty} \frac{2a}{(2\pi n)^2 + a^2} \,. \tag{36}$$

However,

$$\begin{aligned}\sum_{m=-\infty}^{\infty} e^{-a|m|} &= 1 + 2\sum_{m=1}^{\infty} e^{-am} \\ &= 1 + \frac{2e^{-a}}{1-e^{-a}} \\ &= \frac{1+e^{-a}}{1-e^{-a}} \\ &= \frac{e^a+1}{e^a-1} \\ &= \coth(a/2)\,. \end{aligned} \tag{37}$$

Here, I have used the formula $\sum_{n=0}^{\infty} r^n = 1/(1-r)$ for $|r| < 1$. Putting everything together, we have

$$\coth(a/2) = \sum_{n=-\infty}^{\infty} \frac{2a}{(2\pi n)^2 + a^2}\,, \tag{38}$$

as required.

4 Fourier Transform of a Gaussian (is a Gaussian) (15 points)

In this problem, we will show that the Fourier transform of the Gaussian function $f(x) = e^{-ax^2}$ is

$$\tilde{f}(k) = \sqrt{\frac{\pi}{a}} \exp\left(-\frac{k^2}{4a}\right)\,, \tag{39}$$

which is also a Gaussian.

(a) First, given

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} e^{-ax^2}\,, \tag{40}$$

show that the derivative of $\tilde{f}$ with respect to k is

$$\tilde{f}'(k) = -\frac{k}{2a}\tilde{f}(k)\,. \tag{41}$$

SOLUTION:
First, from the definition of $\tilde{f}(k)$, we have

$$\tilde{f}'(k) = \int dx\, (-ix) e^{-ikx} e^{-ax^2}\,. \tag{42}$$

we can evaluate this by integrating by parts, which gives

$$\begin{aligned}\tilde{f}'(k) &= -i\left[-\frac{e^{-ax^2}}{2a}e^{-ikx}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} dx\,(-ik)e^{-ikx}\frac{e^{-ax^2}}{2a}\right] \\ &= -\int_{-\infty}^{\infty}\frac{k}{2a}e^{-ikx}e^{-ax^2} \\ &= -\frac{k}{2a}\tilde{f}(k)\,, \end{aligned} \tag{43}$$

as required.

(b) Thus, show that $\tilde{f} = A\exp(-k^2/4a)$, and that $A = \sqrt{\pi/a}$.

SOLUTION:
From the previous part, we have

$$\frac{d}{dk}\log\tilde{f} = -\frac{k}{2a} \implies \log\tilde{f} = -\frac{k^2}{4a} + C\,, \tag{44}$$

for some constant of integration C. In other words,

$$\tilde{f} = A\exp\left(-\frac{k^2}{4a}\right)\,, \tag{45}$$

for another constant of integration A. However, we know that

$$\tilde{f}(0) = \int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}\,, \tag{46}$$

and so we find

$$\tilde{f} = \sqrt{\frac{\pi}{a}}\exp\left(-\frac{k^2}{4a}\right)\,, \tag{47}$$

as required.

5 Yukawa Potential (30 points)

Consider the Lagrangian for a massive spin-1 field A_μ in Minkowski space (using the mostly-minus metric signature),

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A_\mu A^\mu - A_\mu J^\mu\,, \tag{48}$$

where J_μ is akin to the electromagnetic current.

(a) Find the equations of motion for A_μ. Assuming the current is conserved, i.e. $\partial_\mu J^\mu = 0$, use the equations to find a constraint on A_μ.

SOLUTION:

The equations of motion can be written as

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \tag{49}$$

First, we've done the following calculation already several times:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}\,, \tag{50}$$

while

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = m^2 A^\nu - J^\nu\,. \tag{51}$$

Therefore, the equation of motion is

$$\partial_\mu F^{\mu\nu} = J^\nu - m^2 A^\nu\,. \tag{52}$$

Taking a partial derivative with respect to ∂_ν on both sides, and noting that $\partial_\mu \partial_\nu F^{\mu\nu} = 0$ since $F^{\mu\nu}$ is antisymmetric, and further noting that $\partial_\nu J^\nu = 0$, we must therefore have

$$\partial_\mu A^\mu = 0\,. \tag{53}$$

(b) For $J^\mu \equiv (\rho, \vec{j})$ the 4-current of a stationary point charge $+e$ placed at the origin, show that the equation of motion for A_0 reduces to

$$A^0(r) = \frac{e}{4\pi^2 i r} \int_{-\infty}^{\infty} \frac{dk\, k}{k^2 + m^2} e^{ikr}\,, \tag{54}$$

where r is the distance from the origin. *Hint:* This is a static problem, and so we expect all time derivatives to vanish.

SOLUTION:
The 4-current for a stationary point charge $+e$ is given by $J^0 = e\delta^3(\vec{x})$, and $J^i = 0$. The equation of motion reads

$$\begin{aligned} &\partial_\mu F^{\mu 0} = J^0 - m^2 A^0 \\ &\implies \partial_\mu(\partial^\mu A^0 - \partial^0 A^\mu) + m^2 A^0 = e\delta^3(\vec{x}) \\ &\implies (\partial_\mu \partial^\mu + m^2) A^0 = e\delta^3(\vec{x})\,, \end{aligned} \tag{55}$$

where we have made use of the constraint $\partial_\mu A^\mu = 0$. In addition, time derivatives should vanish, and so we can replace $\partial_\mu \partial^\mu = -\nabla^2$.
Let the 3D spatial Fourier transform of A^0 be $\tilde{A}^0$. Then

$$\begin{aligned} (-\nabla^2 + m^2) A^0 &= (-\nabla^2 + m^2) \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \tilde{A}^0 \\ &= \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} (k^2 + m^2) \tilde{A}^0\,. \end{aligned} \tag{56}$$

On the other hand,

$$e\delta^3(\vec{x}) = e \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}}\,. \tag{57}$$

Therefore, by the uniqueness of the Fourier transform pairs, we can write

$$\tilde{A}^0 = \frac{e}{k^2+m^2} \implies A^0 = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \frac{e}{k^2+m^2} \,. \tag{58}$$

We can now perform the Fourier transform by integrating in spherical coordinates. There is no dependence on the azimuthal angle ϕ, while $\exp(i\vec{k}\cdot\vec{r})$ depends on the polar angle, in particular $y \equiv \cos\theta$. Thus,

$$\begin{aligned} A^0 &= \frac{e}{4\pi^2}\int_0^\infty \frac{dk\,k^2}{k^2+m^2}\int_{-1}^1 dy\, e^{ikry} \\ &= \frac{e}{4\pi^2}\int_0^\infty \frac{dk\,k^2}{k^2+m^2}\frac{e^{ikr}-e^{-ikr}}{ikr} \\ &= \frac{e}{4\pi^2 ir}\int_0^\infty \frac{dk\,k}{k^2+m^2}\left(e^{ikr}-e^{-ikr}\right) \\ &= \frac{e}{4\pi^2 ir}\int_{-\infty}^\infty \frac{dk\,k}{k^2+m^2} e^{ikr} \,, \end{aligned} \tag{59}$$

where the last line can be verified by swapping $k \to -k$ in the second term.

(c) Evaluate this integral with contour integration to get an explicit form for $A^0(r)$. Determine the limit as $m \to 0$. Explain what physical quantity A^0 corresponds to.

SOLUTION:
Consider the function

$$f(z) = \frac{z}{z^2+m^2} e^{izr} \,. \tag{60}$$

This function has poles at $z = \pm im$. Perform the contour integral over C, consisting of *1)* a path along the real line from $-R$ to R and *2)* a semicircle C_R in the upper half plane of radius R, going from R to $-R$. This contour contains one pole at $z = im$, with residue

$$\operatorname*{Res}_{z=im} f(z) = \frac{im}{(im+im)} e^{i(im)r} = \frac{1}{2} e^{-mr} \,. \tag{61}$$

The contour integral over C_R is

$$\int_{C_R} dz\, f(z) = \int_0^\pi d\theta\, iRe^{i\theta} \frac{Re^{i\theta}}{R^2 e^{i2\theta}+m^2} e^{iR\cos\theta} e^{-R\sin\theta} \,, \tag{62}$$

where it should be obvious that the exponential suppression $\exp(-R\sin\theta)$ causes the contour integral to vanish as $R \to \infty$. Therefore, we have

$$\begin{aligned} \int_{-\infty}^\infty \frac{dk\,k}{k^2+m^2} e^{ikr} &= 2\pi i \cdot \frac{1}{2} e^{-mr} = i\pi e^{-mr} \\ \implies A^0 &= \frac{e}{4\pi^2 ir}(i\pi e^{-mr}) \\ &= \frac{e}{4\pi r} e^{-mr} \,. \end{aligned} \tag{63}$$

This potential form is known as the Yukawa potential.
In the limit that $m \to 0$, we have

$$A^0 = \frac{e}{4\pi r} \,, \tag{64}$$

which is the usual electric potential for a positive charge.