Problem Set 7: Complex Analysis I

1 The Other Forms of the Cauchy-Riemann Equations (10 points)

(a) Let $f(z)$ be a differentiable function at some point z_0. Writing $z = re^{i\theta}$, define $f(z) = u(r,\theta) + iv(r,\theta)$, where u and v are real-valued functions. Show that

$$r\partial_r u = \partial_\theta v\,, \quad \partial_\theta u = -r\partial_r v\,. \tag{1}$$

This is the polar form of the Cauchy-Riemann equations.

SOLUTION:
We know that the relation between (x, y) and (r, θ) is given by

$$x = r\cos\theta\,, \quad y = r\sin\theta\,. \tag{2}$$

Thus,

$$\begin{aligned} \partial_r &= \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \cos\theta \cdot \partial_x + \sin\theta \cdot \partial_y\,, \\ \partial_\theta &= \frac{\partial x}{\partial \theta}\partial_x + \frac{\partial y}{\partial \theta}\partial_y = -r\sin\theta \cdot \partial_x + r\cos\theta \cdot \partial_y\,. \end{aligned} \tag{3}$$

The Cauchy-Riemann equations must be satisfied by u and v, since $f(z)$ is differentiable at z_0. But taking derivatives with respect to r and θ, we find

$$\begin{aligned} \partial_r u &= \cos\theta \cdot \partial_x u + \sin\theta \cdot \partial_y u \\ \partial_\theta u &= -r\sin\theta \cdot \partial_x u + r\cos\theta \cdot \partial_y u \\ \partial_r v &= \cos\theta \cdot \partial_x v + \sin\theta \cdot \partial_y v \\ &= -\cos\theta \cdot \partial_y u + \sin\theta \cdot \partial_x u \\ &= -\frac{1}{r}\partial_\theta u \\ \partial_\theta v &= -r\sin\theta \cdot \partial_x v + r\cos\theta \cdot \partial_y v \\ &= r\sin\theta \cdot \partial_y v + r\cos\theta \cdot \partial_x u \\ &= r\partial_r u\,. \end{aligned} \tag{4}$$

Therefore, we find

$$r\partial_r u = \partial_\theta v\,, \quad \partial_\theta u = -r\partial_r v\,, \tag{5}$$

as required.

(b) For a complex number $z = x + iy$, starting from the definitions

$$x = \frac{z + z^*}{2}\,, \quad y = \frac{z - z^*}{2i}\,, \tag{6}$$

by formally applying the chain rule in calculus to some function $F(x, y)$ of two real variables, show that

$$\frac{\partial F}{\partial z^*} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right)\,. \tag{7}$$

This motivates us to define the operator

$$\frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) . \tag{8}$$

Show that if the real and imaginary components of a function $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy-Riemann equations, then

$$\frac{\partial f}{\partial z^*} = 0 . \tag{9}$$

This is the complex form of the Cauchy-Riemann equations.

SOLUTION:
The chain rule gives

$$\begin{aligned}\frac{\partial F}{\partial z^*} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial z^*} \\ &= \frac{1}{2}\frac{\partial F}{\partial x} - \frac{1}{2i}\frac{\partial F}{\partial y} \\ &= \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right) ,\end{aligned} \tag{10}$$

as required.
Next, we note that

$$\begin{aligned}\frac{\partial f}{\partial z^*} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \\ &= \frac{1}{2}\left(\partial_x u + i\partial_x v + i\partial_y u - \partial_y v\right) \\ &= \frac{1}{2}(\partial_x u - \partial_y v) + \frac{i}{2}(\partial_x v - (-\partial_y u)) \\ &= 0 ,\end{aligned} \tag{11}$$

as required.

2 Harmonic Functions (10 points)

(a) Show that if v and V are harmonic conjugates of $u(x, y)$ in a domain D, then $v(x, y)$ and $V(x, y)$ can differ at most by an additive constant.

SOLUTION:
Let v and V be harmonic conjugates of $u(x, y)$ in a domain D. Both functions satisfy the Cauchy-Riemann equations, i.e.

$$\partial_x u = \partial_y v = \partial_y V , \quad \partial_y u = -\partial_x v = -\partial_x V . \tag{12}$$

But this implies that

$$\partial_y(v - V) = 0 , \qquad \partial_x(v - V) = 0 \tag{13}$$

everywhere on the domain D. Therefore, we can conclude that $v - V = C$ for some constant C on the domain D, i.e. $v(x, y)$ and $V(x, y)$ can differ only by an additive constant.

(b) Suppose that v is a harmonic conjugate of u in a domain D, and also that u is a harmonic conjugate of v in D. Show that u and v are both constant functions in D.

SOLUTION:
If v is a harmonic conjugate of u, and also u is a harmonic conjugate of v, we must have the following relations:

$$\begin{aligned} \partial_x u = \partial_y v\,, &\quad \partial_y u = -\partial_x v\,, \\ \partial_x v = \partial_y u\,, &\quad \partial_y v = -\partial_x u\,. \end{aligned} \tag{14}$$

But comparing the two equations, we find $\partial_x u = -\partial_x u$, and $\partial_y u = -\partial_y u$. This is only possible if $\partial_x u = \partial_y u = 0$ everywhere in D, which is true only if u is a constant function.
From this, from the Cauchy-Riemann equations, we can also see that $\partial_x v = \partial_y v = 0$ everywhere in D, and therefore v must also be a constant function.

3 Complex Functions (15 points)

(a) Let z denote any nonzero complex number, written $z = re^{i\Theta}(-\pi < \Theta \leq \pi)$, and let n denote any fixed positive integer. Show that all of the values of $\log(z^{1/n})$ are given by the equation

$$\log(z^{1/n}) = \frac{1}{n}\ln r + i\frac{\Theta + 2(pn+k)\pi}{n}\,, \tag{15}$$

where $p \in \mathbb{Z}$, and $k = 0, 1, 2, \cdots, n-1$. Then, after writing

$$\frac{1}{n}\log z = \frac{1}{n}\ln r + i\frac{\Theta + 2q\pi}{n}\,, \tag{16}$$

where $q \in \mathbb{Z}$, show that the set of values of $\log(z^{1/n})$ is the same as the set of values of $(1/n)\log z$. Thus show that $\log(z^{1/n}) = (1/n)\log z$ where, corresponding to a value of $\log(z^{1/n})$, the appropriate value of $(1/n)\log z$ is to be taken on the right, and conversely.

SOLUTION:
Suppose $z = re^{i\Theta}$ with $-\pi < \Theta \leq \pi$. Then, the n roots of z are given by

$$z^{1/n} = r^{1/n}\exp\left(i\frac{\Theta + 2k\pi}{n}\right), k = 0, 1, 2, \cdots, n-1\,. \tag{17}$$

From this, we find that

$$\begin{aligned} \log(z^{1/n}) &= \ln r^{1/n} + i\left[\frac{\Theta + 2k\pi}{n} + 2p\pi\right] \\ &= \frac{1}{n}\ln r + i\left[\frac{\Theta + 2(pn+k)\pi}{n}\right], p \in \mathbb{Z} \text{ and } k = 0, 1, \cdots, n-1\,, \end{aligned} \tag{18}$$

as required.
On the other hand,

$$\begin{aligned} \frac{1}{n}\log z &= \frac{1}{n}\left[\ln r + i(\Theta + 2q\pi)\right] \\ &= \frac{1}{n}\ln r + i\left(\frac{\Theta + 2q\pi}{n}\right), q \in \mathbb{Z} \end{aligned} \tag{19}$$

Noting that $pn + k$ for $p \in \mathbb{Z}$ and $k = 0, 1, \cdots, n-1$ simply covers all possible integer values, i.e. $pn + k = q$ for some $q \in \mathbb{Z}$, we see that the two expressions are equivalent.

(b) Show that the roots of the equation $\cos z = 2$ are

$$z = 2n\pi \pm i \ln(2 + \sqrt{3}) \,, n \in \mathbb{Z} \,. \tag{20}$$

SOLUTION:
First, from the definition of $\cos z$, we have

$$e^{iz} + e^{-iz} = 4 \,. \tag{21}$$

We can solve for this by substituting $y = e^{iz}$, giving us a quadratic equation $y^2 - 4y + 1 = 0$. The solutions to this are

$$e^{iz} = \frac{1}{2}\left(4 \pm \sqrt{16 - 4}\right) = 2 \pm \sqrt{3} \,. \tag{22}$$

Taking the log of this expression gives

$$i(z + 2n\pi) = \ln(2 \pm \sqrt{3}) \,, n \in \mathbb{Z} \,, \tag{23}$$

Note that

$$2 - \sqrt{3} = \frac{1}{2 + \sqrt{3}} \,, \tag{24}$$

and so we have

$$\begin{aligned} i(z + 2n\pi) &= \pm \ln(2 + \sqrt{3}) \\ z &= 2n\pi \pm i \ln(2 + \sqrt{3}) \,, n \in \mathbb{Z} \,, \end{aligned} \tag{25}$$

as required.

4 Complex Integrals (15 points)

For the functions f and contours C in parts (a) and (b), evaluate

$$\int_C dz \, f(z) \,. \tag{26}$$

(a) $f(z) = \pi \exp(\pi z^*)$, and C is the boundary of the square with vertices at the points 0, 1, $1 + i$, and i, the orientation of C being in the counterclockwise direction.

SOLUTION:
Defining $z = x + iy$, the function can be written as $f(z) = \pi \exp[\pi(x - iy)] = \pi e^{\pi x} e^{-i\pi y}$. We can split the contour into four segments, given by:

(a) *from 0 to 1*: we can write the function as $f(x) = \pi e^{\pi x}$, where $0 \leq x \leq 1$;

(b) *from 1 to* $1 + i$: we can write the function as $f(x) = \pi e^{\pi} e^{-i\pi y}$, where $0 \leq y \leq 1$;

(c) *from* $1 + i$ *to* i: we can write the function as $f(x) = \pi e^{\pi x} e^{-i\pi} = -\pi e^{\pi x}$, where $1 \leq x \leq 0$;

(d) *from* i *to 0*: we can write the function as $f(x) = \pi e^{-i\pi y}$, where $1 \leq y \leq 0$.

Thus,

$$\begin{aligned}\int_C dz\, f(z) &= \int_0^1 dx\,\pi e^{\pi x} + i\int_0^1 dy\,\pi e^{\pi} e^{-i\pi y} - \int_1^0 dx\,\pi e^{\pi x} + i\int_1^0 dy\,\pi e^{-i\pi y} \\ &= e^{\pi x}\big|_0^1 + i\pi e^{\pi}\left.\frac{e^{-i\pi y}}{-i\pi}\right|_0^1 + e^{\pi x}\big|_0^1 - i\pi\left.\frac{e^{-i\pi y}}{-i\pi}\right|_0^1 \\ &= e^{\pi} - 1 - e^{\pi}(-1-1) + e^{\pi} - 1 + (-1-1) \\ &= 4(e^{\pi} - 1) \end{aligned} \tag{27}$$

(b) $f(z)$ is the branch

$$z^{-1+i} = \exp[-(1+i)\log z]\,, \quad (|z| > 0, 0 < \arg(z) < 2\pi) \tag{28}$$

of the indicated power function, and C is the unit circle $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$, oriented counterclockwise.

SOLUTION:
We can write $z = re^{i\theta}$ and the function as

$$\begin{aligned} f(z) &= e^{-(1+i)\log\left(re^{i\theta}\right)} \\ &= e^{-(1+i)(\ln r + i\theta)} \\ &= e^{(1+i)\ln r} e^{(1-i)\theta}\,, 0 < \theta < 2\pi\,. \end{aligned} \tag{29}$$

Along the contour C given by the unit circle $z = e^{i\theta}$, $r = 1$ and $\ln r = 0$. Therefore, the integral is simply

$$\begin{aligned} \int_C dz\, f(z) &= \int_0^{2\pi} d\theta\, f[z(\theta)] z'(\theta) \\ &= \int_0^{2\pi} d\theta\, e^{(1-i)\theta} \cdot ie^{i\theta} \\ &= i\, e^{\theta}\big|_0^{2\pi} \\ &= i(e^{2\pi} - 1)\,. \end{aligned} \tag{30}$$

(c) Let C_0 and C denote the circles $z = z_0 + Re^{i\theta}$ $(-\pi \le \theta \le \pi)$ and $z = Re^{i\theta}$ $(-\pi \le \theta \le \pi)$, respectively. First, show that

$$\int_{C_0} dz\, f(z - z_0) = \int_C dz\, f(z)\,. \tag{31}$$

(Assume that f is a continuous function on C). Use this result to show that

$$\int_{C_0} dz\,(z - z_0)^{n-1} = 0\,, n \in \mathbb{Z}\backslash\{0\}\,, \tag{32}$$

and

$$\int_{C_0} \frac{dz}{z - z_0} = 2\pi i\,. \tag{33}$$

SOLUTION:
First, we see that we can write

$$\int_C dz\, f(z) = \int_{-\pi}^{\pi} d\theta\, f[Re^{i\theta}] \cdot iRe^{i\theta}\,, \tag{34}$$

On the other hand,

$$\begin{aligned}\int_{C_0} dz\, f(z-z_0) &= \int_{-\pi}^{\pi} d\theta\, f[z_0 + Re^{i\theta} - z_0] \cdot iRe^{i\theta} \\ &= \int_{-\pi}^{\pi} d\theta\, f[Re^{i\theta}] \cdot iRe^{i\theta} \\ &= \int_C dz\, f(z)\,, \end{aligned} \tag{35}$$

as required.
From this result, we have

$$\begin{aligned}\int_{C_0} dz\, (z-z_0)^{n-1} &= \int_C dz\, z^{n-1} \\ &= \int_{-\pi}^{\pi} d\theta\, R^{n-1} e^{i(n-1)\theta} \cdot iRe^{i\theta} \\ &= iR^n \int_{-\pi}^{\pi} d\theta\, e^{in\theta} \\ &= \begin{cases} iR^n\, \frac{e^{in\theta}}{in}\Big|_{-\pi}^{\pi} = 0\,, & n \neq 0\,, \\ i(\pi - (-\pi)) = 2\pi i\,, & n = 0\,, \end{cases} \end{aligned} \tag{36}$$

as required.