Problem Set 6: Calculus on Manifolds II

1 Electromagnetism (30 points)

The electromagnetic field strength two form is

$$F = -\frac{1}{2} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,, \tag{1}$$

with coordinates $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z)$, and metric $g = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$ (here, I am using the sloppy notation $\mathrm{d}t^2 \equiv \mathrm{d}t \otimes \mathrm{d}t$ etc., and adopting the mostly minus metric convention, which necessitates the unfortunate minus sign above). In terms of the components of the usual electric and magnetic fields,

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} . \tag{2}$$

Furthermore, let the one-form current $J = \rho\, \mathrm{d}t - j_x\, \mathrm{d}x - j_y\, \mathrm{d}y - j_z\, \mathrm{d}z$, where $\rho, \vec{j}$ are the usual charge density and current density vector respectively. Note that we can write

$$F = -E_x\, \mathrm{d}t \wedge \mathrm{d}x - E_y\, \mathrm{d}t \wedge \mathrm{d}y - E_z\, \mathrm{d}t \wedge \mathrm{d}z + B_z\, \mathrm{d}x \wedge \mathrm{d}y - B_y\, \mathrm{d}x \wedge \mathrm{d}z + B_x\, \mathrm{d}y \wedge \mathrm{d}z \,. \tag{3}$$

(a) Show that $\mathrm{d}F = 0$ is equivalent to the two homogeneous Maxwell equations. Notice that if we introduce A such that $F = \mathrm{d}A$, then the homogeneous Maxwell equations are automatically satisfied.

SOLUTION:
We have

$$\mathrm{d}F = -\frac{1}{2} \partial_\alpha F_{\mu\nu}\, \mathrm{d}x^\alpha \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = 0 \,. \tag{4}$$

There are four independent components in this sum, namely $\{\alpha, \mu, \nu\} = \{0,1,2\}, \{0,1,3\}, \{0,2,3\}, \{1,2,3\}$, and so each of these components have to separately be zero. Let's look at the $\{0,1,2\}$ combination first. There are six terms, but e.g.

$$\frac{1}{2} \partial_0 F_{12}\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \partial_0 F_{21}\, \mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^1 = \partial_0 F_{12}\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \tag{5}$$

Therefore, the components must satisfy

$$\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = 0 \implies \partial_t B_z + \partial_x E_y - \partial_y E_x = 0 \,. \tag{6}$$

The other components are similarly

$$\begin{aligned} \partial_0 F_{23} + \partial_2 F_{30} + \partial_3 F_{02} = 0 &\implies \partial_t B_x + \partial_y E_z - \partial_z E_y = 0 \,, \\ \partial_0 F_{13} + \partial_1 F_{30} + \partial_3 F_{01} = 0 &\implies -\partial_t B_y + \partial_x E_z - \partial_z E_x = 0 \,, \\ \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0 &\implies \partial_x B_x + \partial_y B_y + \partial_z B_z = 0 \,. \end{aligned} \tag{7}$$

The last expression is clearly $\nabla \cdot \vec{B} = 0$, while the other three are equivalent to $\nabla \cdot \vec{E} = -\partial_t \vec{B}$, which are the two homogeneous Maxwell equations.

(b) Show that $\mathrm{d} \star F = \star J$ is equivalent to the two inhomogeneous Maxwell equations.

SOLUTION: First, let's begin by calculating the Hodge dual of J:

$$\star J = \rho \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z - j_x \, \mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}z + j_y \, \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}z - j_z \, \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \,. \tag{8}$$

On other other hand,

$$\star F = E_x \, \mathrm{d}y \wedge \mathrm{d}z - E_y \, \mathrm{d}x \wedge \mathrm{d}z + E_z \mathrm{d}x \wedge \mathrm{d}y + B_z \, \mathrm{d}t \wedge \mathrm{d}z + B_y \, \mathrm{d}t \wedge \mathrm{d}y + B_x \, \mathrm{d}t \wedge \mathrm{d}x \,. \tag{9}$$

and therefore

$$\begin{aligned} \mathrm{d} \star F =& (\partial_t E_x - \partial_y B_z + \partial_z B_y) \mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}z + (\partial_x E_x + \partial_y E_y + \partial_z E_z) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &+ (-\partial_t E_y - \partial_x B_z + \partial_z B_x) \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}z + (\partial_t E_z - \partial_x B_y + \partial_y B_x) \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \,. \end{aligned} \tag{10}$$

Equating this with $\star J$ gives

$$\begin{aligned} \partial_x E_x + \partial_x E_y + \partial_z E_z &= \rho \,, \\ \partial_t E_x - \partial_y B_z + \partial_z B_y &= -j_x \,, \\ -\partial_t E_y - \partial_x B_z + \partial_z B_x &= j_y \,, \\ \partial_t E_z - \partial_x B_y + \partial_y B_x &= -j_z \,. \end{aligned} \tag{11}$$

The first equation reads $\nabla \cdot \vec{E} = \rho$, while the remaining three equations can be combined into $\nabla \times \vec{B} = \vec{j} + \partial_t \vec{E}$, which are the two inhomogeneous Maxwell equations.

(c) Show that $\mathrm{d} \star J = \mathrm{d}(\mathrm{d} \star F) = 0$ is equivalent to the continuity equation for charge $\partial_t \rho + \nabla \cdot \vec{j} = 0$. Thus, the conservation of charge follows directly from the inhomogeneous Maxwell equations.

SOLUTION:
From our previous results, we have immediately that

$$\begin{aligned} \mathrm{d} \star J =& \partial_t \rho \, \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z - \partial_x j_x \, \mathrm{d}x \wedge \mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &+ \partial_y j_y \, \mathrm{d}y \wedge \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}z - \partial_z j_z \, \mathrm{d}z \wedge \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \\ =& (\partial_t \rho + \partial_x j_x + \partial_y j_y + \partial_z j_z) \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \,. \end{aligned} \tag{12}$$

$\mathrm{d} \star J = \mathrm{d}(\mathrm{d} \star F) = 0$ therefore implies that

$$\partial_t \rho + \nabla \cdot \vec{j} = 0 \,, \tag{13}$$

as required.

(d) Defining $A = A_\mu \, \mathrm{d}x^\mu = \phi \, \mathrm{d}t - A_x \, \mathrm{d}x - A_y \, \mathrm{d}y - A_z \, \mathrm{d}z$, where ϕ and $\vec{A}$ are the usual scalar and vector potential, show that $F = -\mathrm{d}A$ leads to $\vec{E} = -\partial_t \vec{A} - \nabla \phi$ and $\vec{B} = \nabla \times \vec{A}$.

SOLUTION:
We have, component by component,

$$\begin{aligned} \mathrm{d}A &= (-\partial_x \phi - \partial_t A_x) \, \mathrm{d}t \wedge \mathrm{d}x + (-\partial_y \phi - \partial_t A_y) \, \mathrm{d}t \wedge \mathrm{d}y + (-\partial_z \phi - \partial_t A_z) \, \mathrm{d}t \wedge \mathrm{d}z \\ &= (\partial_y A_x - \partial_x A_y) \, \mathrm{d}x \wedge \mathrm{d}y + (\partial_z A_x - \partial_x A_z) \, \mathrm{d}x \wedge \mathrm{d}z + (\partial_z A_y - \partial_y A_z) \, \mathrm{d}y \wedge \mathrm{d}z \,. \end{aligned} \tag{14}$$

Comparing this with $-F$ gives

$$E_x = -\partial_x \phi - \partial_t A_x \quad E_y = -\partial_y \phi - \partial_t A_y \quad E_z = -\partial_z \phi - \partial_t A_z \tag{15}$$

$$-B_z = \partial_y A_x - \partial_x A_y \quad B_y = \partial_z A_x - \partial_x A_z \quad -B_x = \partial_z A_y - \partial_y A_z \,. \tag{16}$$

which is precisely $\vec{E} = -\partial_t \vec{A} - \nabla\phi$ and $\vec{B} = \nabla \times \vec{A}$.

(e) Show that the gauge transformation $A \to A + \mathrm{d}\chi$ for arbitrary smooth χ leaves the field strength F invariant.

SOLUTION:
Under the gauge transformation $A \to A + \mathrm{d}\chi$, we have

$$F \mapsto -\mathrm{d}(A + \mathrm{d}\chi) = -\mathrm{d}A - \mathrm{d}^2\chi = -\mathrm{d}A = F\,, \tag{17}$$

and hence the field strength tensor is invariant.

(f) The action for the electromagnetic field on Minkowski spacetime is a local functional of the potential A. There are two 4-forms that we can construct for the field strength $F = -\mathrm{d}A$, so the simplest possible action is

$$S[A] = \int_{\mathbb{R}^{1,3}} (\alpha F \wedge \star F + \theta F \wedge F)\ , \tag{18}$$

where $\alpha, \theta \in \mathbb{R}$ are coefficients. Rewrite this action in terms of $\vec{E}$ and $\vec{B}$.

SOLUTION:
Repeating our earlier results, we find

$$\begin{aligned} F &= -E_x\,\mathrm{d}t \wedge \mathrm{d}x - E_y\,\mathrm{d}t \wedge \mathrm{d}y - E_z\,\mathrm{d}t \wedge \mathrm{d}z + B_z\,\mathrm{d}x \wedge \mathrm{d}y - B_y\,\mathrm{d}x \wedge \mathrm{d}z + B_x\,\mathrm{d}y \wedge \mathrm{d}z\,, \\ \star F &= E_x\,\mathrm{d}y \wedge \mathrm{d}z - E_y\,\mathrm{d}x \wedge \mathrm{d}z + E_z \mathrm{d}x \wedge \mathrm{d}y + B_z\,\mathrm{d}t \wedge \mathrm{d}z + B_y\,\mathrm{d}t \wedge \mathrm{d}y + B_x\,\mathrm{d}t \wedge \mathrm{d}x\,. \end{aligned} \tag{19}$$

Performing the wedge product, we find

$$\begin{aligned} F \wedge \star F &= (-E_x^2 - E_y^2 - E_z^2 + B_z^2 + B_y^2 + B_x^2)\,\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &= (\vec{B}^2 - \vec{E}^2)\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\,. \end{aligned} \tag{20}$$

On the other hand,

$$\begin{aligned} F \wedge F &= 2(-E_x B_x - E_y B_y - E_z B_z)\,\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &= -2(\vec{E} \cdot \vec{B})\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\,. \end{aligned} \tag{21}$$

Therefore, the action becomes

$$S[\vec{E}, \vec{B}] = \int_{\mathbb{R}^{1,3}} \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \left(\alpha(\vec{B}^2 - \vec{E}^2) - 2\theta \vec{E} \cdot \vec{B}\right)\ . \tag{22}$$

(g) Return to the action given in Eq. (18), find the equations of motion for A by varying $A \mapsto A + \delta A$, and assuming that $A \to 0$ at infinity. What role does θ play?

SOLUTION:
Under the variation $A \mapsto A + \delta A$, we have

$$F \mapsto -\mathrm{d}(A + \delta A) = F - \mathrm{d}(\delta A)\,, \tag{23}$$

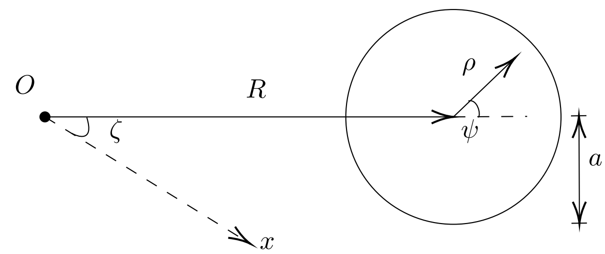

Figure 1: A parametrization of the torus.

from which we see first that

$$\begin{aligned} F \wedge F \mapsto &(F - \mathrm{d}(\delta A)) \wedge (F - \mathrm{d}(\delta A)) \\ &= F \wedge F - \mathrm{d}(\delta A) \wedge F - F \wedge \mathrm{d}(\delta A) \\ &= F \wedge F - 2\mathrm{d}(\delta A) \wedge F\,, \end{aligned} \tag{24}$$

where in the last line I have used the fact that $\mathrm{d}(\delta A) \wedge F = F \wedge \mathrm{d}(\delta A)$, since $\mathrm{d}(\delta A)$ and F are both 2-forms. However, we now also have

$$\mathrm{d}(\delta A) \wedge F = \mathrm{d}(\delta A \wedge F)\,, \tag{25}$$

since $\mathrm{d}F = \mathrm{d}^2 A = 0$. Next, we have

$$\begin{aligned} F \wedge \star F \mapsto &(F - \mathrm{d}(\delta A)) \wedge \star(F - \mathrm{d}(\delta A)) \\ &= F \wedge \star F - \mathrm{d}(\delta A) \wedge \star F - F \wedge \star \mathrm{d}(\delta A) \\ &= F \wedge \star F - 2\mathrm{d}(\delta A) \wedge \star F \\ &= F \wedge \star F - 2\mathrm{d}(\delta A \wedge \star F) - 2\delta A \wedge \mathrm{d} \star F\,, \end{aligned} \tag{26}$$

by a very similar set of arguments as above, except that $\mathrm{d} \star F \neq 0$, and in the last line we used $\mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^p \omega \wedge (\mathrm{d}\eta)$, for ω a p-form.
Hence, under the variation,

$$\delta S = \int_{\mathbb{R}^{1,3}} \alpha \left[-2\mathrm{d}(\delta A \wedge \star F) - 2\delta A \wedge \mathrm{d} \star F\right] + 2\theta \left[2\mathrm{d}(\delta A \wedge F)\right]\,. \tag{27}$$

However, we can drop the total derivative terms, since by Stokes' theorem, they simply become an integral over the boundary at infinity. This leaves

$$\delta S = -2\alpha \int_{\mathbb{R}^{1,3}} (\delta A \wedge \mathrm{d} \star F) \tag{28}$$

for an arbitrary variation δA. Therefore, we must have $\mathrm{d} \star F = 0$, which is simply the inhomogeneous Maxwell equation with no sources, as should have been anticipated. θ did not play any role here, since it is proportional to a term that is a total derivative.

2 The Torus (20 points)

Fig. 1 shows a cross-section of a solid torus embedded in $\mathbb{R}^3$ with major radius R and minor radius a, taking a cut through the torus at some angle ζ with respect to the x-axis of $\mathbb{R}^3$. A convenient parametrization for the solid torus is also shown.

(a) Show that in terms of the parametrization shown, the volume element is

$$\mathrm{d}\omega = \rho(R + \rho\cos\psi)\mathrm{d}\rho \wedge \mathrm{d}\zeta \wedge \mathrm{d}\psi\,. \tag{29}$$

SOLUTION:
The relation between Cartesian coordinates and the parametrization shown is

$$\begin{aligned} x &= (R + \rho\cos\psi)\cos\zeta\,, \\ y &= (R + \rho\cos\psi)\sin\zeta\,, \\ z &= \rho\sin\psi\,. \end{aligned} \tag{30}$$

The metric induced on the solid torus is then

$$g_{ij} = \sum_a \frac{\partial x^a}{\partial \xi^i}\frac{\partial x^a}{\partial \xi^j}\,, \tag{31}$$

where $\xi \equiv (\rho, \psi, \zeta)$, and a runs over the Cartesian coordinates. Working through everything, we find

$$\begin{aligned} g_{\rho\rho} &= (\cos\psi\cos\zeta)^2 + (\cos\psi\sin\zeta)^2 + (\sin\psi)^2 = 1\,, \\ g_{\rho\zeta} &= (\cos\psi\cos\zeta)(-(R+\rho\cos\psi)\sin\zeta) + (\cos\psi\sin\zeta)((R+\rho\cos\psi)\cos\zeta) = 0\,, \\ g_{\rho\psi} &= (\cos\psi\cos\zeta)(-\rho\sin\psi\cos\zeta) + (\cos\psi\sin\zeta)(-\rho\sin\psi\sin\zeta) + \rho\sin\psi\cos\psi = 0\,, \\ g_{\zeta\zeta} &= (-(R+\rho\cos\psi)\sin\zeta)^2 + ((R+\rho\cos\psi)\cos\zeta)^2 = (R+\rho\cos\psi)^2\,, \\ g_{\zeta\psi} &= (-\rho\sin\psi\cos\zeta)(-(R+\rho\cos\psi)\sin\zeta) + (-\rho\sin\psi\sin\zeta)((R+\rho\cos\psi)\cos\zeta) = 0\,, \\ g_{\psi\psi} &= (-\rho\sin\psi\cos\zeta)^2 + (-\rho\sin\psi\sin\zeta)^2 + (\rho\cos\psi)^2 = \rho^2\,. \end{aligned} \tag{32}$$

Therefore,

$$\sqrt{g} = \rho(R + \rho\cos\psi)\,, \tag{33}$$

and the volume element is

$$\mathrm{d}\omega = \rho(R + \rho\cos\psi)\mathrm{d}\rho \wedge \mathrm{d}\zeta \wedge \mathrm{d}\psi\,. \tag{34}$$

You can check that this choice of the orientation agrees with the choice of conventional choice of orientation in $\mathbb{R}^3$ by setting $\zeta = 0$, and noting that $\hat{\rho} = \hat{x}$, $\hat{\zeta} = \hat{y}$ and $\hat{\psi} = \hat{z}$.

(b) Calculate the total volume V of the solid torus by performing a direct integral over the volume form of the solid torus.

SOLUTION:
The total volume of the solid torus is therefore

$$V = \int_0^{2\pi} \mathrm{d}\zeta \int_0^{2\pi} \mathrm{d}\psi \int_0^a \mathrm{d}\rho\, \rho(R + \rho\cos\psi) = 4\pi^2 R\frac{a^2}{2} = 2\pi^2 R a^2\,, \tag{35}$$

since $\cos\psi$ integrates to zero over a full period.

(c) Calculate the total volume V of the solid torus by using Stoke's theorem and converting the integral to one over the surface of the torus. Check that the result agrees with the previous part.

SOLUTION:
By Stoke's theorem,

$$\int_T \mathrm{d}\omega = \int_{\partial T} \omega \,, \tag{36}$$

where T and ∂T are the solid torus and the boundary of the torus respectively. Therefore, We can see that

$$\omega = \left(\frac{\rho^2}{2} R + \frac{\rho^2}{3} \cos\psi \right) \mathrm{d}\zeta \wedge \mathrm{d}\psi \,. \tag{37}$$

Integrating this over the boundary of the torus gives

$$V = \int_0^{2\pi} \mathrm{d}\zeta \int_0^{2\pi} \mathrm{d}\psi \left(\frac{a^2}{2} R + \frac{a^2}{3} \cos\psi \right) = 2\pi^2 a^2 R \,, \tag{38}$$

dropping the $\cos\psi$ term since it integrates to zero over a full period. This agrees with the previous result.

(d) Calculate the total surface area A of the torus by deducing the appropriate area form and integrating appropriately.

SOLUTION:
The area form for the 2D surface of the torus can be obtained directly from the previous calculation, by noting that the induced metric is just the 2×2 submatrix of the full metric containing the ψ and ζ components, setting $\rho = a$. The area form is therefore

$$d\eta = a(R + a\cos\psi)\mathrm{d}\zeta \wedge \mathrm{d}\psi \,. \tag{39}$$

Integrating over the torus gives

$$A = \int_0^{2\pi} d\zeta \int_0^{2\pi} d\psi \, a(R + a\cos\psi) = 4\pi^2 a R \,, \tag{40}$$

where we once again drop the $\cos\psi$ term since it integrates to zero over a full period.

3 Complex Analysis Warm-Up (10 points)

(a) Prove the triangle inequality for complex numbers,

$$|z_1 + z_2| \leq |z_1| + |z_2| \tag{41}$$

for any $z_1, z_2 \in \mathbb{C}$.

SOLUTION:
First, we see that

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)^*(z_1 + z_2) \\ &= (z_1^* + z_2^*)(z_1 + z_2) \\ &= |z_1|^2 + z_1^* z_2 + z_2^* z_1 + |z_2|^2 \,, \end{aligned} \tag{42}$$

However, $z_1^* z_2 + z_2^* z_1 = 2\mathrm{Re}(z_1^* z_2) \leq 2|z_1^* z_2|$, since

$$\mathrm{Re}(z_1^* z_2) \leq \sqrt{[\mathrm{Re}(z_1^* z_2)]^2 + [\mathrm{Im}(z_1^* z_2)]^2} = |z_1^* z_2| \,. \tag{43}$$

Thus,

$$|z_1 + z_2|^2 \leq |z_1|^2 + 2|z_1^* z_2| + |z_2|^2 = |z_1|^2 + 2|z_1||z_2| + |z_2|^2 = (|z_1| + |z_2|)^2 \,, \tag{44}$$

or

$$|z_1 + z_2| \leq |z_1| + |z_2| \,. \tag{45}$$

(b) Prove the following: for $z \in \mathbb{C}$, $z \neq 1$,

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \,. \tag{46}$$

Use this to derive Lagrange's trigonometric identity, which for $\theta \neq 2\pi k$, $k \in \mathbb{Z}$, reads

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \,. \tag{47}$$

SOLUTION:
Denote $S = 1 + z + z^2 + \cdots + z^n$. Then,

$$zS = z + z^2 + z^3 + \cdots + z^{n+1} \,, \tag{48}$$

and

$$S - zS = 1 - z^{n+1} \implies S = \frac{1 - z^{n+1}}{1 - z} \,, \tag{49}$$

as required.
Taking $z = e^{i\theta}$, we have

$$\begin{aligned} 1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} &= \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \\ &= \frac{e^{i(n+1)\theta/2}}{e^{i\theta/2}} \frac{e^{-i(n+1)\theta/2} - e^{i(n+1)\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}} \\ &= e^{in\theta/2} \frac{-2i\sin[(n+1)\theta/2]}{-2i\sin(\theta/2)} \\ &= \frac{\sin[(n+1)\theta/2]}{\sin(\theta/2)} e^{in\theta/2} \end{aligned} \tag{50}$$

Now, taking the real part on both sides gives

$$\begin{aligned} 1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta &= \frac{\sin[(n+1)\theta/2]}{\sin(\theta/2)} \cos(n\theta/2) \\ &= \frac{1}{2\sin(\theta/2)} \left(2\sin[(n+1)\theta/2]\cos(n\theta/2)\right) \\ &= \frac{1}{2\sin(\theta/2)} \left(\sin[(2n+1)\theta/2] + \sin(\theta/2)\right) \\ &= \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \,, \end{aligned} \tag{51}$$

as required.

(c) Sketch the following sets of points: i) $|z-1+i|=1$, ii) $|z-1|=|z+i|$, and iii) $|2z^*+i|\leq 4$. Please label your sketches clearly with key features (radii, intercepts, etc.). If these are lines, please specify their gradients (i.e. the usual $\Delta y/\Delta x$ where the y-axis should be taken to be the imaginary axis, and the x-axis the real axis).

SOLUTION:
i) This is a circle of radius 1 centered on $1-i$.
ii) This is a line passing through the origin, gradient -1.
iii) We have $|2z^*+i|=|2z-i|$, and so this is equivalent to $|z-i/2|\leq 2$ is a filled circle centered on $i/2$, radius 2.