Problem Set 5: Calculus on Manifolds I

1 The Gradient (5 points)

Consider the function on $f : \mathbb{R}^3 \to \mathbb{R}$ defined in Cartesian coordinates by $f(x, y, z) = x^2 + y^2 + z^2$.

(a) Compute the gradient $\mathrm{d}f = a_i \mathrm{d}x^i$ in Cartesian coordinates, i.e. find expressions for a_i.

> **SOLUTION:**
> The gradient is
>
> $$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z = 2x\,\mathrm{d}x + 2y\,\mathrm{d}y + 2z\,\mathrm{d}z\,. \tag{1}$$

(b) Evaluate $\mathrm{d}f(V)$ for $V = x\partial_x - y\partial_y + \partial_z$.

> **SOLUTION:**
> Given that $\mathrm{d}x^\mu(\partial_\nu) = \delta^\mu_\nu$, we have
>
> $$\mathrm{d}f(V) = 2x^2 - 2y^2 + 2z\,. \tag{2}$$

(c) Express $\mathrm{d}f$ in spherical coordinates (r, θ, ϕ).

> **SOLUTION:**
> We can see immediately that $f(x, y, z) = r^2$, and so without any further computation, we have $\mathrm{d}f = 2r\,\mathrm{d}r$.

(d) Evaluate $\mathrm{d}f(V)$ for $V = \partial_r - \partial_\theta + \partial_\phi$.

> **SOLUTION:** As before, $\mathrm{d}x^\mu(\partial_\nu) = \delta^\mu_\nu$, and therefore $\mathrm{d}f(V) = 2r$.

2 Properties of the Lie Bracket (10 points)

Prove the following properties of the Lie bracket between two vector fields X and Y:

(a) The Lie bracket is linear, i.e. for functions f, g on M and a, b real numbers, $[X, Y](af + bg) = a[X, Y]f + b[X, Y]g$.

> **SOLUTION:** Starting from the definition of the Lie bracket, we have
>
> $$[X, Y](f) = X(Y(f)) - Y(X(f))\,. \tag{3}$$
>
> From the linearity of X and Y (since they are simply directional derivatives),
>
> $$[X, Y](af + bg) = X(Y(af + bg)) - Y(X(af + bg)) = X(aY(f) + bY(g)) - Y(aX(f) + bX(g))\,. \tag{4}$$

Now applying linearity again,

$$\begin{aligned}[X,Y](af+bg) &= aX(Y(f)) + bX(Y(g)) - aY(X(f)) - bY(X(g)) \\ &= a[X,Y](f) + b[X,Y](g)\,, \end{aligned} \tag{5}$$

as required.

(b) The Lie bracket obeys Leibniz's rule, i.e. $[X,Y](fg) = f[X,Y]g + g[X,Y]f$.

SOLUTION: Again, remembering that X and Y are simply directional derivatives, and therefore individually satisfy Leibniz's rule, we have

$$[X,Y](fg) = X(Y(fg)) - Y(X(fg)) = X(fY(g) + gY(f)) - Y(fX(g) + gX(f))\,. \tag{6}$$

The object $X(g)$ is simply taking the directional derivative at every point on the manifold of the function g, and so is yet another scalar function. We can therefore apply Leibniz's rule for each of X and Y to get

$$\begin{aligned}[X,Y](fg) =& X(f)Y(g) + fX(Y(g)) + X(g)Y(f) + gX(Y(f)) \\ & - Y(f)X(g) - fY(X(g)) - Y(g)X(f) - gY(X(f)) \\ =& f[X(Y(g)) - Y(X(g))] + g[X(Y(f)) - Y(X(f))] \\ =& f[X,Y]g + g[X,Y]f\,, \end{aligned} \tag{7}$$

as required.

(c) In terms of the components of $X \equiv X^\mu \partial_\mu$ and $Y \equiv Y^\mu \partial_\mu$, the Lie bracket is given by $[X,Y]^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu$.

SOLUTION: In terms of components, we see that

$$[X,Y] = X^\mu \partial_\mu Y^\nu \partial_\nu - Y^\mu \partial_\mu X^\nu \partial_\nu = (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)\partial_\nu\,. \tag{8}$$

and therefore

$$[X,Y]^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu\,, \tag{9}$$

as required.

(d) $[X,Y]^\mu$ transforms under a change of coordinates as a vector.

SOLUTION: Under a change of coordinates, we see that

$$\begin{aligned} X^\nu \partial_\nu Y^\mu \mapsto & X'^\alpha \partial'_\alpha Y'^\beta \\ &= \frac{\partial x'^\alpha}{\partial x^\nu} X^\nu \frac{\partial x^\nu}{\partial x'^\alpha} \partial_\nu \left(\frac{\partial x'^\beta}{\partial x^\mu} Y^\mu \right) \\ &= X^\nu \left[\frac{\partial x'^\beta}{\partial x^\mu} \partial_\nu Y^\mu + Y^\mu \frac{\partial x'^\beta}{\partial x^\mu \partial x^\nu} \right]\,. \end{aligned} \tag{10}$$

Similarly, we have

$$Y^\nu \partial_\nu X^\mu \mapsto Y^\nu \left[\frac{\partial x'^\beta}{\partial x^\mu} \partial_\nu X^\mu + X^\mu \frac{\partial x'^\beta}{\partial x^\mu \partial x^\nu} \right] , \tag{11}$$

which we can obtain easily be just swapping $X \leftrightarrow Y$. Therefore,

$$\begin{aligned} X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu \mapsto X^\nu \left[\frac{\partial x'^\beta}{\partial x^\mu} \partial_\nu Y^\mu + Y^\mu \frac{\partial x'^\beta}{\partial x^\mu \partial x^\nu} \right] - Y^\nu \left[\frac{\partial x'^\beta}{\partial x^\mu} \partial_\nu X^\mu + X^\mu \frac{\partial x'^\beta}{\partial x^\mu \partial x^\nu} \right] \\ = X^\nu \left[\frac{\partial x'^\beta}{\partial x^\mu} \partial_\nu Y^\mu + Y^\mu \frac{\partial x'^\beta}{\partial x^\mu \partial x^\nu} \right] - Y^\mu \left[\frac{\partial x'^\beta}{\partial x^\nu} \partial_\mu X^\nu + X^\nu \frac{\partial x'^\beta}{\partial x^\nu \partial x^\mu} \right] \\ = \frac{\partial x'^\beta}{\partial x^\mu} (X^\nu \partial_\nu Y^\mu - Y^\mu \partial_\mu X^\nu) , \end{aligned} \tag{12}$$

which therefore transforms as a vector. The pesky part which also was the reason why the partial derivative of a vector isn't a tensor drops out. Note that in the second to last line, I am simply relabeling $\mu \leftrightarrow \nu$, which I can do because the indices are summed over and are dummy indices. I also used the fact that partial derivatives commute (at least in physics).

3 The Lie Derivative of the Metric (10 points)

Prove that the Lie derivative of the metric is

$$(\mathcal{L}_X g)_{\mu\nu} = X^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha} \partial_\nu X^\alpha + g_{\alpha\nu} \partial_\mu X^\alpha . \tag{13}$$

SOLUTION: The Lie derivative is defined as follows: for vector fields X and Y, as well as a scalar field f, we have $\mathcal{L}_X f = X(f)$, and $\mathcal{L}_X Y = [X, Y]$. Furthermore, the Lie derivative obeys linearity and Leibniz's rule.

With this in mind, let W and Y be vector fields on the manifold. Consider the object $g_{\mu\nu} W^\mu Y^\nu$, which is a scalar. The Lie derivative acting on a scalar should give

$$\begin{aligned} \mathcal{L}_X (g_{\mu\nu} W^\mu Y^\nu) &= X^\alpha \partial_\alpha (g_{\mu\nu} W^\mu Y^\nu) \\ &= W^\mu Y^\nu X^\alpha \partial_\alpha g_{\mu\nu} + X^\alpha g_{\mu\nu} \partial_\alpha (W^\mu Y^\nu) . \end{aligned} \tag{14}$$

On the other hand, applying Leibniz's rule,

$$\begin{aligned} \mathcal{L}_X (g_{\mu\nu} W^\mu Y^\nu) &= \mathcal{L}_X g_{\mu\nu} \cdot W^\mu Y^\nu + g_{\mu\nu} \mathcal{L}_X (W^\mu Y^\nu) \\ &= \mathcal{L}_X g_{\mu\nu} \cdot W^\mu Y^\nu + g_{\mu\nu} (Y^\nu \mathcal{L}_X W^\mu + W^\mu \mathcal{L}_X Y^\nu) . \end{aligned} \tag{15}$$

But

$$\begin{aligned} Y^\nu \mathcal{L}_X W^\mu + W^\mu \mathcal{L}_X Y^\nu &= Y^\nu (X^\alpha \partial_\alpha W^\mu - W^\alpha \partial_\alpha X^\mu) + W^\mu (X^\alpha \partial_\alpha Y^\nu - Y^\alpha \partial_\alpha X^\nu) \\ &= X^\alpha \partial_\alpha (W^\mu Y^\nu) - (W^\alpha \partial_\alpha X^\mu) Y^\nu - (Y^\alpha \partial_\alpha X^\nu) W^\mu \end{aligned} \tag{16}$$

Therefore, comparing Eqs. (14) and (15), we have

$$\begin{aligned} \mathcal{L}_X g_{\mu\nu} \cdot W^\mu Y^\nu &= W^\mu Y^\nu X^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\nu} [(W^\alpha \partial_\alpha X^\mu) Y^\nu + (Y^\alpha \partial_\alpha X^\nu) W^\mu] \\ &= W^\mu Y^\nu X^\alpha \partial_\alpha g_{\mu\nu} + W^\mu Y^\nu g_{\alpha\nu} \partial_\mu X^\alpha + W^\mu Y^\nu g_{\mu\alpha} \partial_\nu X^\alpha , \end{aligned} \tag{17}$$

where we have done some relabeling of the dummy indices in the last line. We can therefore conclude that

$$\mathcal{L}_X g_{\mu\nu} = X^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha} \partial_\nu X^\alpha + g_{\alpha\nu} \partial_\mu X^\alpha , \tag{18}$$

as required.

4 (SG 11.2) Rigid Rotations on the Sphere (10 points)

The metric on the unit sphere in polar coordinates is

$$g = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \, \mathrm{d}\phi \otimes \mathrm{d}\phi \,. \tag{19}$$

Consider

$$V_x = -\sin\phi \, \partial_\theta - \cot\theta \cos\phi \, \partial_\phi \,, \tag{20}$$

which is a vector field of a rigid rotation about the x-axis. Show that $\mathcal{L}_{V_x} g = 0$, i.e. that rigid rotations do not deform the metric.

SOLUTION: For convenience, let me just rename V_x as V. Then the Lie derivative of the metric is

$$(\mathcal{L}_V g)_{\mu\nu} = V^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha} \partial_\nu V^\alpha + g_{\alpha\nu} \partial_\mu V^\alpha \,. \tag{21}$$

Component by component, noting that the derivative of $g_{\theta\theta}$ in any coordinate is zero, since it is a constant, we have

$$\begin{aligned} (\mathcal{L}_V g)_{\theta\theta} &= 2 g_{\theta\theta} \partial_\theta V^\theta \\ &= 0 \,, \end{aligned} \tag{22}$$

since V^θ only depends on ϕ. Next,

$$\begin{aligned} (\mathcal{L}_V g)_{\theta\phi} &= g_{\theta\theta} \partial_\phi V^\theta + g_{\phi\phi} \partial_\theta V^\phi \\ &= -\cos\phi + \sin^2\theta \cdot (\csc^2\theta \, \cos\phi) \\ &= 0 \,. \end{aligned} \tag{23}$$

$(\mathcal{L}_V g)_{\phi\theta} = (\mathcal{L}_V g)_{\theta\phi} = 0$ by symmetry (you can check by permuting the indices yourself). Finally,

$$\begin{aligned} (\mathcal{L}_V g)_{\phi\phi} &= V^\theta \partial_\theta g_{\phi\phi} + 2 g_{\phi\phi} \partial_\phi V^\phi \\ &= -\sin\phi (2 \sin\theta \cos\theta) + 2 \sin^2\theta (\cot\theta \sin\phi) \\ &= 0 \,. \end{aligned} \tag{24}$$

Therefore $\mathcal{L}_{V_x} g = 0$, as required.

5 Wedge Product (10 points)

Consider two one-forms λ, η on $\mathbb{R}^2$, defined in Cartesian coordinates by

$$\lambda = 2xy \, \mathrm{d}x + (x^2 + y^2) \mathrm{d}y \,, \quad \eta = e^{xy} \mathrm{d}x - \mathrm{d}y \,. \tag{25}$$

(a) Compute the wedge product $\lambda \wedge \eta$.

SOLUTION: The wedge product is

$$\begin{aligned}\lambda \wedge \eta &= (2xy\,\mathrm{d}x + (x^2+y^2)\mathrm{d}y) \wedge (e^{xy}\mathrm{d}x - \mathrm{d}y) \\ &= -2xy\,\mathrm{d}x \wedge \mathrm{d}y + (x^2+y^2)e^{xy}\,\mathrm{d}y \wedge \mathrm{d}x \\ &= -(2xy + (x^2+y^2)e^{xy})\,\mathrm{d}x \wedge \mathrm{d}y\,. \end{aligned} \tag{26}$$

(b) Compute $\lambda \wedge \eta(V, W)$ where $V = x\partial_x + y\partial_y$ and $W = \partial_x - \partial_y$.

SOLUTION: We have

$$\begin{aligned}\lambda \wedge \eta(V,W) &= -(2xy + (x^2+y^2)e^{xy})\,\mathrm{d}x \wedge \mathrm{d}y(V,W) \\ &= -(2xy + (x^2+y^2)e^{xy})(-x-y) \\ &= (x+y)(2xy + (x^2+y^2)e^{xy})\,. \end{aligned} \tag{27}$$

(c) Express $\lambda \wedge \eta$ in polar coordinates (ρ, θ).

SOLUTION: Switching to polar coordinates, we have $x = \rho\cos\theta$, $y = \rho\sin\theta$, and

$$\begin{aligned} xy &= \rho^2 \cos\theta\,\sin\theta\,, \\ (x^2+y^2)e^{xy} &= \rho^2 e^{\rho^2\cos\theta\sin\theta}\,. \end{aligned} \tag{28}$$

Furthermore,

$$\begin{aligned}\mathrm{d}x \wedge \mathrm{d}y &= \left(\frac{\partial x}{\partial \rho}\mathrm{d}\rho + \frac{\partial x}{\partial \theta}\mathrm{d}\theta\right) \wedge \left(\frac{\partial y}{\partial \rho}\mathrm{d}\rho + \frac{\partial y}{\partial \theta}\mathrm{d}\theta\right) \\ &= \rho\cos^2\theta\,\mathrm{d}\rho \wedge \mathrm{d}\theta + (-\rho\sin\theta)(\sin\theta)\mathrm{d}\theta \wedge \mathrm{d}\rho \\ &= \rho(\sin^2\theta + \cos^2\theta)\mathrm{d}\rho \wedge \mathrm{d}\theta \\ &= \rho\,\mathrm{d}\rho \wedge \mathrm{d}\theta\,. \end{aligned} \tag{29}$$

Thus,

$$\lambda \wedge \eta = -(2\rho^2\cos\theta\sin\theta + \rho^2 e^{\rho^2\cos\theta\sin\theta})\,\rho\,\mathrm{d}\rho \wedge \mathrm{d}\theta\,. \tag{30}$$

6 Exterior Derivatives (10 points)

Compute the exterior derivatives of the following forms:

(a) $f = xy$.

SOLUTION:

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y = y\,\mathrm{d}x + x\,\mathrm{d}y\,. \tag{31}$$

(b) $A = y\,\mathrm{d}x + xy\,\mathrm{d}y$.

SOLUTION:

$$\begin{aligned} \mathrm{d}A &= \mathrm{d}y \wedge \mathrm{d}x + (y\,\mathrm{d}x + x\,\mathrm{d}y) \wedge \mathrm{d}y \\ &= (y-1)\mathrm{d}x \wedge \mathrm{d}y\,. \end{aligned} \tag{32}$$

(c) $F = yx\,\mathrm{d}x \wedge \mathrm{d}y + xz\,\mathrm{d}y \wedge \mathrm{d}z$.

SOLUTION:

$$\begin{aligned} \mathrm{d}F &= (y\,\mathrm{d}x + x\,\mathrm{d}y) \wedge \mathrm{d}x \wedge \mathrm{d}y + (x\,\mathrm{d}z + z\,\mathrm{d}x) \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &= z\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\,. \end{aligned} \tag{33}$$

(d) $\omega = xyz(\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}z \wedge \mathrm{d}x)$.

SOLUTION:

$$\begin{aligned} \mathrm{d}\omega &= (yz\,\mathrm{d}x + xz\,\mathrm{d}y + xy\,\mathrm{d}z) \wedge (\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}z \wedge \mathrm{d}x) \\ &= (xy + yz + xz)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\,. \end{aligned} \tag{34}$$