Problem Set 4: Tensors

1 Dual Basis (5 points)

Let

$$\vec{e}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} , \qquad \vec{e}_2 = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{1}$$

be a basis for $\mathbb{R}^2$, where the column vectors correspond to their components in the standard basis, denoted by $\hat{x}_i$. Find the basis covectors $\{\vec{e}^{*1}, \vec{e}^{*2}\}$, which are dual to $\{\vec{e}_1, \vec{e}_2\}$, written in terms of the usual dual basis $\hat{x}^{*i}$, where $\hat{x}^{*i}(\hat{x}_j) = \delta^i_j$.

SOLUTION:
We know that

$$\vec{e}^{*i}(\vec{e}_j) = \delta^i_j \tag{2}$$

Write $\vec{e}_i = a^j{}_i \hat{x}_j$, and $\vec{e}^{*i} = f^i{}_j \hat{x}^{*j}$, where $f^i{}_j$ are the components of $\vec{e}^{*i}$ in the standard basis. By definition,

$$\vec{e}^{*i}(\vec{e}_j) = \delta^i_j \,. \tag{3}$$

But we also have

$$\vec{e}^{*i}(\vec{e}_j) = f^i{}_l \hat{x}^{*l}(a^k{}_j \hat{x}_k) = f^i{}_l a^k{}_j \delta^l_k = f^i{}_k a^k{}_j \,. \tag{4}$$

Therefore,

$$f^i{}_k a^k{}_j = \delta^i_j \implies f^i{}_k = (a^{-1})^i{}_k \,. \tag{5}$$

But as a matrix,

$$a = \begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix} \implies a^{-1} = \frac{1}{6} \begin{pmatrix} 5 & -4 \\ -1 & 2 \end{pmatrix} \tag{6}$$

Thus,

$$\vec{e}^{*1} = \frac{5}{6}\hat{x}^{*1} - \frac{2}{3}\hat{x}^{*2} \qquad \vec{e}^{*2} = -\frac{1}{6}\hat{x}^{*1} + \frac{1}{3}\hat{x}^{*2} \,. \tag{7}$$

2 (SG 10.10) Quotient Theorem (5 points)

Suppose that you have come up with some recipe for generating an array of numbers T^{ijk} in any coordinate frame, and want to know whether these numbers are the components of a triply contravariant tensor. Suppose further that you know that, given the components of a_{ij} of an arbitrary doubly covariant tensor, the numbers

$$T^{ijk} a_{jk} = v^i \tag{8}$$

transforms as the components of a contravariant vector. Show that T^{ijk} does indeed transform as a triply contravariant tensor. (The natural generalization of this result to arbitrary tensor types is known as the **quotient theorem**.)

SOLUTION:
Under a change of basis $\vec{e}'_j = \alpha^i{}_j \vec{e}_i$, we have

$$a_{jk} = \alpha^{j'}{}_j \alpha^{k'}{}_k a_{j'k'} \,, \qquad v^i = (\alpha^{-1})^i{}_{i'} v^{i'} \,. \tag{9}$$

Therefore, we have

$$\begin{aligned} T^{ijk}\alpha^{j'}{}_j \alpha^{k'}{}_k a_{j'k'} = (\alpha^{-1})^i{}_{m'} v^{m'} &\implies \alpha^{i'}{}_i T^{ijk}\alpha^{j'}{}_j \alpha^{k'}{}_k a_{j'k'} = \alpha^{i'}{}_i (\alpha^{-1})^i{}_{m'} v^{m'} \\ &\implies \alpha^{i'}{}_i \alpha^{j'}{}_j \alpha^{k'}{}_k T^{ijk} a_{j'k'} = \delta^{i'}_{m'} v^{m'} = v^{i'} \end{aligned} \tag{10}$$

Defining $T^{i'j'k'} = \alpha^{i'}{}_i \alpha^{j'}{}_j \alpha^{k'}{}_k T^{ijk}$, we see that $T^{i'j'k'} a_{j'k'} = v^{i'}$, and that therefore $T^{i'j'k'}$ transforms as a triply covariant tensor, as required.

3 (SG 10.11) Invariant Content of a (1,1)-Tensor (10 points)

Let $T^i{}_j$ be the 3×3 array of components of a tensor. Consider the objects

$$a = T^i{}_i \,, \qquad b = T^i{}_j T^j{}_i \,, \qquad c = T^i{}_j T^j{}_k T^k{}_i \,. \tag{11}$$

Show explicitly that c is an invariant. Describe these objects in terms of properties of the matrix $T^i{}_j$.

Assume that $T^i{}_j$ has 3 distinct eigenvalues. Show that the eigenvalues of the linear map represented by T can be found by solving the equation

$$\lambda^3 - a\lambda^2 + \frac{1}{2}(a^2 - b)\lambda - \frac{1}{6}(a^3 - 3ab + 2c) = 0 \,. \tag{12}$$

Hint: Choose a good basis!

SOLUTION:
Under a change of basis,

$$\begin{aligned} T^i{}_j T^j{}_k T^k{}_i &\mapsto a^{i'}{}_i (a^{-1})^j{}_{j'} T^i{}_j \, a^{j'}{}_l (a^{-1})^m{}_{k'} T^l{}_m \, a^{k'}{}_n (a^{-1})^p{}_{i'} T^n{}_p \\ &= a^{i'}{}_i (a^{-1})^j{}_{j'} a^{j'}{}_l (a^{-1})^m{}_{k'} a^{k'}{}_n (a^{-1})^p{}_{i'} T^i{}_j T^l{}_m T^n{}_p \\ &= \delta^j_l \delta^m_n \delta^p_i T^i{}_j T^l{}_m T^n{}_p \\ &= T^i{}_j T^j{}_k T^k{}_i \,, \end{aligned} \tag{13}$$

as required. In terms of the matrix $T^i{}_j$, we have $a = \mathrm{Tr}(T)$, $b = \mathrm{Tr}(T^2)$, and $c = \mathrm{Tr}(T^3)$.
The eigenvalues of the linear map represented by T are the roots of the characteristic equation

$$\det(T - \lambda I) = 0 \tag{14}$$

for T in any basis. Since the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are distinct, T is diagonalizable, and hence we can choose the eigenbasis where $T^i{}_j = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. Therefore, the eigenvalues satisfy the equation

$$(\lambda - \tau_1)(\lambda - \tau_2)(\lambda - \tau_3) = 0 \tag{15}$$

where I have simplified the notation so that $\tau_i = T^i{}_i$, with *no summation being implied here*. Also in this basis,

$$a = \tau_1 + \tau_2 + \tau_3 \,, \quad b = \tau_1^2 + \tau_2^2 + \tau_3^2 \,, \quad c = \tau_1^3 + \tau_2^3 + \tau_3^3 \,. \tag{16}$$

From these relations, we can see that

$$\tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1 = \frac{1}{2}\left[(\tau_1 + \tau_2 + \tau_3)^2 - (\tau_1^2 + \tau_2^2 + \tau_3^2)\right] = \frac{1}{2}(a^2 - b) \,, \tag{17}$$

and

$$6\tau_1\tau_2\tau_3 = (\tau_1+\tau_2+\tau_3)^3 - (\tau_1^3+\tau_2^3+\tau_3^3) - 3(\tau_1+\tau_2+\tau_3)(\tau_1^2+\tau_2^2+\tau_3^2) + 3(\tau_1^3+\tau_2^3+\tau_3^3) = a^3 + 2c - 3ab \quad (18)$$

Thus, we finally have

$$(\lambda-\tau_1)(\lambda-\tau_2)(\lambda-\tau_3) = \lambda^3 - \lambda^2 a + \frac{1}{2}(a^2-b)\lambda - \frac{1}{6}(a^3-3ab+2c) = 0\,. \quad (19)$$

as required.

4 (SG 10.15) Symmetric Integration (15 points)

Show that the n-dimensional integral of

$$I_{\alpha\beta\gamma\delta} = \int \frac{d^n k}{(2\pi)^n}\, k_\alpha k_\beta k_\gamma k_\delta f(k^2) \quad (20)$$

is given by

$$I_{\alpha\beta\gamma\delta} = A(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})\,, \quad (21)$$

where

$$A = \frac{1}{n(n+2)} \int \frac{d^n k}{(2\pi)^n} k^4 f(k^2)\,. \quad (22)$$

Here, $\vec{k}$ is a regular vector in $\mathbb{R}^n$.

Similarly evaluate

$$I_{\alpha\beta\gamma\delta\epsilon} = \int \frac{d^n k}{(2\pi)^n}\, k_\alpha k_\beta k_\gamma k_\delta k_\epsilon f(k^2)\,. \quad (23)$$

SOLUTION: To solve this, we want to show that $I_{\alpha\beta\gamma\delta}$ is invariant under $O(n)$ transformations. Under an orthogonal transformation O, we see that

$$I_{\alpha\beta\gamma\delta} \mapsto O^{\alpha}{}_{\alpha'} O^{\beta}{}_{\beta'} O^{\gamma}{}_{\gamma'} O^{\delta}{}_{\delta'} I_{\alpha\beta\gamma\delta} = \int \frac{d^n k}{(2\pi)^n} O^{\alpha}{}_{\alpha'} k_\alpha O^{\beta}{}_{\beta'} k_\beta O^{\gamma}{}_{\gamma'} k_\gamma O^{\delta}{}_{\delta'} k_\delta f(k^2)\,. \quad (24)$$

Let's relabel $\tilde{k}_{\alpha'} = O^{\alpha}{}_{\alpha'} k_\alpha$, noting that $\tilde{k}^2 = k^2$, since the transformation is orthogonal. We can also perform a change of variables in the integral, with

$$d^n k = \left|\frac{\partial \vec{k}}{\partial \tilde{\vec{k}}}\right| d^n \tilde{k}\,, \quad (25)$$

where $\left|\partial\vec{k}/\partial\tilde{\vec{k}}\right|$ is the absolute value of the Jacobian associated with the transformation. Since $k_\alpha = (O^{-1})^{\alpha'}{}_{\alpha} \tilde{k}_{\alpha'}$, and O^{-1} is orthogonal, we have $d^n k = d^n \tilde{k}$. Putting this altogether, we find

$$I_{\alpha\beta\gamma\delta} \mapsto \int \frac{d^n \tilde{k}}{(2\pi)^n} \tilde{k}_\alpha \tilde{k}_\beta \tilde{k}_\gamma \tilde{k}_\delta f(\tilde{k}^2) = I_{\alpha\beta\gamma\delta}\,, \quad (26)$$

since we can simply relabel again $\tilde{k} \to k$.
We therefore conclude that $I_{\alpha\beta\gamma\delta}$ is invariant under $O(n)$, and can be written as

$$I_{\alpha\beta\gamma\delta} = A\delta_{\alpha\beta}\delta_{\gamma\delta} + B\delta_{\alpha\gamma}\delta_{\beta\delta} + C\delta_{\alpha\delta}\delta_{\beta\gamma} \,. \tag{27}$$

First, note that under the swap of any index, e.g. $\alpha \leftrightarrow \beta$, $I_{\alpha\beta\gamma\delta} = I_{\beta\alpha\gamma\delta}$, and so symmetry enforces $A = B = C$. Now, let's contract α and β, as well as γ and δ. We get

$$\begin{aligned} I^{\alpha}{}_{\alpha}{}^{\gamma}{}_{\gamma} = \delta^{\alpha\beta}\delta^{\gamma\delta}I_{\alpha\beta\gamma\delta} &= A\delta^{\alpha\beta}\delta^{\gamma\delta}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \\ &= A(\delta^{\alpha\beta}\delta_{\alpha\beta}\delta^{\gamma\delta}\delta_{\gamma\delta} + \delta^{\alpha\beta}\delta_{\alpha\gamma}\delta^{\gamma\delta}\delta_{\beta\delta} + \delta^{\alpha\beta}\delta_{\alpha\delta}\delta^{\gamma\delta}\delta_{\beta\gamma}) \\ &= A(n^2 + \delta^{\beta}_{\gamma}\delta^{\gamma}_{\beta} + \delta^{\beta}_{\delta}\delta^{\delta}_{\beta}) \\ &= A(n^2 + n + n) = An(n+2) \,. \end{aligned} \tag{28}$$

On the other hand,

$$I^{\alpha}{}_{\alpha}{}^{\gamma}{}_{\gamma} = \int \frac{d^n k}{(2\pi)^n} k^4 f(k^2) \,, \tag{29}$$

and therefore

$$A = \frac{1}{n(n+2)} \int \frac{d^n k}{(2\pi)^n} k^4 f(k^2) \,, \tag{30}$$

as required.

5 (SG 10.5) Properties of the Levi-Civita Symbol (10 points)

We defined the n-dimensional Levi-Civita symbol by requiring that $\epsilon_{i_1 i_2 \cdots i_n}$ be antisymmetric under the swapping of any pair of indices, and $\epsilon_{12\cdots n} = 1$.

(a) Show that if any two indices $i_i = i_j$ for $i, j \in 1, \cdots, n$, then $\epsilon_{i_1 i_2 \cdots i_n} = 0$.

SOLUTION:
Suppose $i_i = i_j$. Then swapping $i_i \leftrightarrow i_j$ picks up a relative minus sign, but the Levi-Civita symbol remains unchanged. Anything which is the negative of itself must be zero.

(b) Show that $\epsilon_{123} = \epsilon_{231} = \epsilon_{312}$, but that $\epsilon_{1234} = -\epsilon_{2341} = \epsilon_{3412} = -\epsilon_{4123}$.

SOLUTION:
ϵ_{231} can be obtained from ϵ_{123} by swapping $1 \leftrightarrow 2$ followed by $1 \leftrightarrow 3$. Since this is two swaps, there is no relative sign. Similarly, ϵ_{312} can be obtained from ϵ_{123} by swapping $1 \leftrightarrow 3$ followed by $1 \leftrightarrow 2$. Thus, $\epsilon_{231} = \epsilon_{312} = \epsilon_{123}$.
Similarly, to get ϵ_{2341} from ϵ_{1234}, we need to swap $1 \leftrightarrow 2$, $2 \leftrightarrow 3$, and $3 \leftrightarrow 4$. This is three swaps, and so we get a relative sign of -1. The other relations can be shown similarly.

(c) Show that

$$\epsilon_{ijk}\epsilon_{i'j'k'} = \delta^{i}_{i'}\delta^{j}_{j'}\delta^{k}_{k'} + 5 \text{ other terms} \,, \tag{31}$$

where you should write out all six terms explicitly.

SOLUTION:

$$\epsilon_{ijk}\epsilon_{i'j'k'} = \delta_i^{i'}\delta_j^{j'}\delta_k^{k'} + \delta_i^{j'}\delta_j^{k'}\delta_k^{i'} + \delta_i^{k'}\delta_j^{i'}\delta_k^{j'} - \delta_i^{i'}\delta_j^{k'}\delta_k^{j'} - \delta_i^{j'}\delta_j^{i'}\delta_k^{k'} - \delta_i^{k'}\delta_j^{j'}\delta_k^{i'} \,. \tag{32}$$

(d) Show that $\epsilon_{ijk}\epsilon_{ij'k'} = \delta^j_{j'}\delta^k_{k'} - \delta^j_{k'}\delta^k_{j'}$.

SOLUTION:

$$\begin{aligned}\epsilon_{ijk}\epsilon_{ij'k'} &= \delta_i^{i'}\epsilon_{ijk}\epsilon_{i'j'k'} \\ &= 3\delta_j^{j'}\delta_k^{k'} + 2\delta_j^{k'}\delta_k^{j'} - 3\delta_j^{k'}\delta_k^{j'} - 2\delta_j^{j'}\delta_k^{k'} \\ &= \delta_j^{j'}\delta_k^{k'} - \delta_j^{k'}\delta_k^{j'} \,,\end{aligned} \tag{33}$$

as required.

(e) For dimension $n = 4$, write out $\epsilon_{ijkl}\epsilon_{ij'k'l'}$ as a sum of products of Kronecker deltas similar to the one in part (c).

SOLUTION:
We can easily see that in general,

$$\epsilon_{ijkl}\epsilon_{i'j'k'l'} = \text{sgn}(i'j'k'l')\delta_i^{i'}\delta_j^{j'}\delta_k^{k'}\delta_l^{l'} + \text{remaining 23 permutations of } i'j'k'l' \,, \tag{34}$$

where sgn is positive for even permutations of $i'j'k'l'$ and negative for odd permutations. But we can group these permutations as

$$\begin{aligned}\epsilon_{ijkl}\epsilon_{i'j'k'l'} =& \delta_i^{i'}\left(\text{sgn}(i',j'k'l')\delta_j^{j'}\delta_k^{k'}\delta_l^{l'} + \text{remaining 5 permutations of } j'k'l'\right) \\ &+ \delta_i^{j'}\left(\text{sgn}(j',i'k'l')\delta_j^{i'}\delta_k^{k'}\delta_l^{l'} + \text{remaining 5 permutations of } i'k'l'\right) \\ &+ \delta_i^{k'}\left(\text{sgn}(k',j'i'l')\delta_j^{j'}\delta_k^{i'}\delta_l^{l'} + \text{remaining 5 permutations of } j'i'l'\right) \\ &+ \delta_i^{l'}\left(\text{sgn}(l',j'k'i')\delta_j^{j'}\delta_k^{k'}\delta_l^{i'} + \text{remaining 5 permutations of } j'k'i'\right)\end{aligned} \tag{35}$$

Multiplying both sides by $\delta^i_{i'}$, we get

$$\begin{aligned}\epsilon_{ijkl}\epsilon_{ij'k'l'} &= (4-1-1-1)\left(\text{sgn}(i',j'k'l')\delta_j^{j'}\delta_k^{k'}\delta_l^{l'} + \text{remaining 5 permutations of } j'k'l'\right) \\ &= \delta_i^{i'}\delta_j^{j'}\delta_k^{k'} + \delta_i^{j'}\delta_j^{k'}\delta_k^{i'} + \delta_i^{k'}\delta_j^{i'}\delta_k^{j'} - \delta_i^{i'}\delta_j^{k'}\delta_k^{j'} - \delta_i^{j'}\delta_j^{i'}\delta_k^{k'} - \delta_i^{k'}\delta_j^{j'}\delta_k^{i'} \\ &= \epsilon_{jkl}\epsilon_{j'k'l'} \,.\end{aligned} \tag{36}$$