Problem Set 3: Calculus of Variations III

1 SG 1.10: Piano String (15 points)

A piano string can vibrate both transversely and longitudinally, but is fixed at its two ends. Transverse vibrations cause local compressions in the string and thus the two kinds of vibrations can influence each other. A Lagrangian that takes into account the lowest order effect of stretching on the local string tension is

$$L[\xi,\eta] = \int dx \left\{ \frac{1}{2}\rho_0 \left[(\partial_t \xi)^2 + (\partial_t \eta)^2\right] - \frac{\lambda}{2}\left[\frac{\tau_0}{\lambda} + \partial_x \xi + \frac{1}{2}(\partial_x \eta)^2\right]^2 \right\}, \tag{1}$$

where $\xi(x,t)$ denotes the longitudinal displacement and $\eta(x,t)$ denotes the transverse displacement of the string. That is, the point in the undisturbed string which had coordinates $(x,0)$ is moved to the point $(x+\xi(x,t), \eta(x,t))$. The parameter τ_0 represents the tension in the undisturbed string, λ is the product of the Young's modulus and cross-sectional area, and ρ_0 is the mass per unit length.

a) Extremize the action to derive the coupled equations of motion for ξ and η.

SOLUTION: We can directly use the Euler-Lagrange equations since we are only interested in variations with fixed endpoints. This reads (switching to notation where overdot denotes a time derivative and $'$ denotes a spatial derivative)

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \xi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \xi)} = 0 &\implies \rho_0 \ddot{\xi} - \lambda \partial_x \left[\frac{\tau_0}{\lambda} + \xi' + \frac{1}{2}\eta'^2\right] = 0 \\ &\implies \rho_0 \ddot{\xi} - \lambda \xi'' - \lambda \eta' \eta'' = 0\,, \end{aligned} \tag{2}$$

where $\mathcal{L}$ is the Lagrangian density, as well as

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \eta} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \eta)} = 0 &\implies \rho_0 \ddot{\eta} - \lambda \partial_x \left[\left(\frac{\tau_0}{\lambda} + \xi' + \frac{1}{2}\eta'^2\right)\eta'\right] = 0 \\ &\implies \rho_0 \ddot{\eta} - \lambda \left[(\xi'' + \eta'\eta'')\eta' + \left(\frac{\tau_0}{\lambda} + \xi' + \frac{1}{2}\eta'^2\right)\eta''\right] = 0 \\ &\implies \rho_0 \ddot{\eta} - \tau_0 \eta'' - \lambda\left(\xi''\eta' + \xi'\eta'' + \frac{3}{2}\eta'^2\eta''\right) = 0 \end{aligned} \tag{3}$$

b) Is the canonical energy-momentum tensor $T^\nu{}_\mu$ conserved? Find $T^\nu{}_\mu$, keeping terms up to quadratic order in ξ, η and their derivatives.

SOLUTION:
Yes it is: the Lagrangian density has no explicit dependence on ξ and η, and therefore exhibits space and time translation symmetry. At leading order in the fields, the Lagrangian density may be written as

$$\mathcal{L} \approx \frac{1}{2}\rho_0\left(\dot{\xi}^2 + \dot{\eta}^2\right) - \frac{\lambda}{2}\left(\frac{\tau_0}{\lambda} + \xi'\right)^2 - \frac{\tau_0}{2}\eta'^2\,. \tag{4}$$

The canonical stress-energy tensor is given by

$$T^\nu{}_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \xi)}\partial_\mu \xi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \eta)}\partial_\mu \eta - \delta^\nu{}_\mu \mathcal{L}\,. \tag{5}$$

Term by term, we have

$$\begin{aligned} T^0{}_0 &= \rho_0\dot{\xi}^2 + \rho_0\dot{\eta}^2 - \frac{1}{2}\rho_0\left(\dot{\xi}^2 + \dot{\eta}^2\right) + \frac{\lambda}{2}\left(\frac{\tau_0}{\lambda} + \xi'\right)^2 + \frac{\tau_0}{2}\eta'^2 \\ &= \frac{1}{2}\rho_0\left(\dot{\xi}^2 + \dot{\eta}^2\right) + \frac{\lambda}{2}\left(\frac{\tau_0}{\lambda} + \xi'\right)^2 + \frac{\tau_0}{2}\eta'^2\,, \qquad (6) \\ T^0{}_1 &= \rho_0(\dot{\xi}\xi' + \dot{\eta}\eta')\,, \qquad (7) \\ T^1{}_0 &= -\lambda\left(\frac{\tau_0}{\lambda} + \xi'\right)\dot{\xi} - \tau_0\eta'\dot{\eta}\,, \qquad (8) \\ T^1{}_1 &= -\lambda\left(\frac{\tau_0}{\lambda} + \xi'\right)\xi' - \frac{1}{2}\rho_0\left(\dot{\xi}^2 + \dot{\eta}^2\right) + \frac{\lambda}{2}\left(\frac{\tau_0}{\lambda} + \xi'\right)^2 + \frac{\tau_0}{2}\eta'^2 \\ &= -\frac{1}{2}\rho_0\left(\dot{\xi}^2 + \dot{\eta}^2\right) - \frac{\lambda}{2}\left(\frac{\tau_0}{\lambda} + \xi'\right)\left(\frac{\tau_0}{\lambda} - \xi'\right) + \frac{\tau_0}{2}\eta'^2\,. \qquad (9) \end{aligned}$$

c) Work in the limit where ξ, η and their derivatives are all considered small. Show that at leading order, the longitudinal and transverse motions decouple. Find expressions for the longitudinal and transverse wave velocities c_L and c_T in terms of the material parameters. Show that $\eta = \eta_0(x - c_T t)$ and $\xi = 0$ is a solution to the equations of motion at leading order.

SOLUTION:
The linearized equations are

$$\begin{aligned} \rho_0\ddot{\xi} - \lambda\xi'' &= 0\,, \\ \rho_0\ddot{\eta} - \tau_0\eta'' &= 0\,, \qquad (10) \end{aligned}$$

which are wave equations for the longitudinal and transverse modes respectively, with $c_L = \sqrt{\lambda/\rho_0}$ and $c_T = \sqrt{\tau_0/\rho_0}$.
$\xi = 0$ is trivially a solution to the wave equation for ξ. On the other hand, if we define $y = x - c_T t$, then

$$\eta_0'' = \frac{d^2\xi}{dy^2}\,, \qquad w\ddot{\eta}_0 = c_T^2\frac{d^2\eta}{dy^2}\,, \qquad (11)$$

and so we can see that $\ddot{\eta}_0 - c_T^2\eta_0'' = 0$, i.e. η_0 is a solution to the equation of motion for η to leading order.

d) Write down the equations of motion at next-to-leading order. Given the leading order solution $\eta = \eta_0(x - c_T t)$ and $\xi = 0$, show that the next-to-leading order solution for ξ is of the form $\xi = \xi_0(x - c_T t)$, i.e. it is only a function of $x - c_T t$.

SOLUTION: This now reads

$$\begin{aligned} \rho_0\ddot{\xi} - \lambda\xi'' &= \lambda\eta'\eta'' \\ \rho_0\ddot{\eta} - \tau_0\eta'' &= \lambda\left(\xi''\eta' + \xi'\eta''\right) \qquad (12) \end{aligned}$$

Suppose $\xi = \xi_0(x - c_T t)$, then

$$\rho_0\ddot{\xi} - \lambda\xi'' = \rho_0\frac{d^2\xi_0}{dy^2}\left(c_T^2 - c_L^2\right) \qquad (13)$$

But $\lambda\eta'\eta''$ is also just a function of $x - c_T t$, and so both sides of the expression are consistent, as long as we choose ξ_0 appropriately, i.e.

$$\rho_0(c_T^2 - c_L^2)\frac{d^2\xi_0}{dy^2} = \lambda\eta'\eta'' = c_L^2\rho_0\frac{d\eta_0}{dy}\frac{d^2\eta_0}{dy^2} \implies \frac{d^2\xi_0}{dy^2} = \frac{1}{2}\frac{c_L^2}{c_T^2 - c_L^2}\frac{d}{dy}\eta_0'^2 \,. \tag{14}$$

e) Show that ξ_0 must satisfy

$$\rho_0\dot{\xi}_0 = \frac{1}{2}\frac{c_L^2}{c_L^2 - c_T^2}T^0{}_1 \,. \tag{15}$$

You may assume that we are considering an infinitely long string, and therefore that all solutions and their derivatives must go to zero at infinity.

SOLUTION: We can now integrate the equation above to get

$$\frac{d\xi_0}{dy} = \frac{1}{2}\frac{c_L^2}{c_T^2 - c_L^2}\eta_0'^2 + A \,, \tag{16}$$

Based on the requirement that all derivatives must vanish at infinity, we can safely set $A = 0$. Looking at $T^0{}_1$, we have

$$T^0{}_1 = \rho_0\left(\dot{\xi}_0\xi_0' + \dot{\eta}_0\eta_0'\right) \approx \rho_0\dot{\eta}_0\eta_0' = -\rho_0 c_T\eta_0'^2 \,, \tag{17}$$

since ξ is higher order than η (remember that ξ is nonzero only because η is nonzero). Therefore

$$\frac{d\xi_0}{dy} = \frac{1}{-c_T}\dot{\xi}_0 = \frac{1}{2}\frac{c_L^2}{c_T^2 - c_L^2}\frac{T^0{}_1}{-\rho_0 c_T} \implies \rho_0\dot{\xi}_0 = \frac{1}{2}\frac{c_L^2}{c_T^2 - c_L^2}T^0{}_1 \,. \tag{18}$$

2 SG 1.6: The Catenary Revisited (10 points)

Let's revisit the catenary problem using an intrinsic parametrization in terms of $(x(s), y(s))$ where s is the arc-length. For a chain of length L, the potential energy is then $U = \int_0^L ds\,\rho g y(s)$, but $x(s)$ and $y(s)$ are not independent functions of s because they must satisfy the constraint $\dot{x}^2 + \dot{y}^2 = 1$ at every point on the curve (i.e. the speed of a particle traveling on the curve in terms of the arc-length is unity). Refer to Fig. 1 for the geometry of the set-up throughout.

a) Introduce infinitely many Lagrange multipliers $\lambda(s)$ (i.e. use a function as a Lagrange multiplier) to enforce the arc-length constraint at every point s. From the resulting functional F, derive the coupled Euler-Lagrange equations for $x(s)$ and $y(s)$. By considering the forces acting on a small section ds of the hanging cable, show that $\lambda(s)$ is proportional to the position dependent tension $T(s)$ in the chain. (*Hint*: you may wish to introduce the angle $\psi(s)$ of the curve, where $\dot{x} = \cos\psi$ and $\dot{y} = \sin\psi$.)

SOLUTION: The functional we want to minimize is

$$\tilde{U}(x, y, \lambda) = \int_0^L ds\,\left[\rho g y(s) - \lambda(s)\left(\dot{x}^2 + \dot{y}^2 - 1\right)\right] \,. \tag{19}$$

The equation of motion for λ enforces the length constraint. With no endpoint variation, we can simply use the Euler-Lagrange equation for x and y to get

$$\frac{d}{ds}[\lambda(s)\dot{x}] = 0 \,, \qquad \rho g + 2\frac{d}{ds}[\lambda(s)\dot{y}] = 0 \,. \tag{20}$$

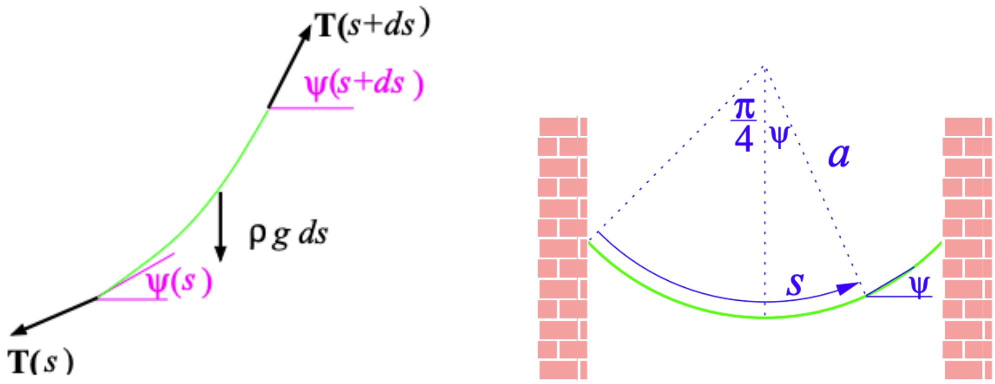

Figure 1: (Left) The geometry of the tension and weight acting on a piece of string of length ds. (Right) The subtended circular arc. Figures taken from SG.

Consider a piece of the chain of length ds located between positions s and $s+ds$, with the geometry shown in Fig. 1. The horizontal component of the tension force at s is simply $T_x = T(s)\cos\psi$, and the vertical component $T_y = T(s)\sin\psi$, using the definition in the hint. As we go across this piece of the chain, horizontal component becomes $T_x + dT_x$, and the vertical $T_y + dT_y$. Since the chain is in equilibrium, the horizontal components of the tension must cancel, i.e.

$$d(T\cos\psi) = 0 \implies \frac{d}{ds}(T\dot{x}) = 0\,, \tag{21}$$

which is identical to the Euler-Lagrange equation for x. On the other hand, the increase in the vertical component dT_y must balance the weight $\rho g\, ds$, giving

$$d(T\sin\psi) = \rho g\, ds \implies \rho g - \frac{d}{ds}(T\dot{y}) = 0\,. \tag{22}$$

Thus, we recover the Euler-Lagrange equations by setting $\lambda(s) = -T(s)/2$, i.e. $\lambda(s)$ is proportional to $T(s)$, as required.

b) You have a lightweight cord of length $\pi a/2$ and some lead shot of total mass M. How should you distribute the mass along the line in order that the cord hangs in a circular arc subtending an angle of $\pi/2$ when its ends are attached to two hooks at the same height? You can use the equations in part *a)*, adapted to allow a position dependent $\rho(s)$, to find $\rho(s)$ that produces such a shape.

SOLUTION:
Using the results above, we can see that

$$\frac{d}{ds}[\lambda(s)\cos\psi] = 0\,, \qquad \rho g + 2\frac{d}{ds}[\lambda(s)\sin\psi] = 0\,. \tag{23}$$

The first equation says that we can define a constant $C = \lambda(s)\cos\psi$, and so the second reads

$$\rho g = -2C\frac{d}{ds}\tan\psi\,. \tag{24}$$

However, the geometry of the set-up shown in Fig. 1 also shows that $s/a + \pi/4 + \psi = \pi/2$, taking $0 \leq s \leq \pi a/2$. Thus,

$$\rho g = -2C \frac{d}{ds} \tan\left(\frac{\pi}{4} - \frac{s}{a}\right) = \frac{2}{a} C \sec^2\left(\frac{\pi}{4} - \frac{s}{a}\right) \tag{25}$$

To find the constant, we integrate both sides to get

$$\begin{aligned} Mg &= \int_0^{\pi a/2} ds\, \frac{2C}{a} \sec^2\left(\frac{\pi}{4} - \frac{s}{a}\right) \\ &= 2C \tan\left(\frac{\pi}{4} - \frac{s}{a}\right)\Big|_{\pi a/2}^{0} \\ &= 4C \implies C = \frac{Mg}{4}\,. \end{aligned} \tag{26}$$

Thus, we have

$$\rho = \frac{M}{2a} \sec^2\left(\frac{\pi}{4} - \frac{s}{a}\right)\,. \tag{27}$$

3 (SG1.5 and SG10.14): Elastic Waves (20 points)

Throughout this problem, we work with Cartesian coordinates where we need not differentiate between lower and upper indices. Suppose an elastic body Ω of density ρ is slightly deformed so that the point that was at Cartesian coordinates x^i is moved to $x^i + \eta^i(t, x)$. Here, η is called the displacement field. We define the Cartesian strain tensor e_{ij} by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial \eta_j}{\partial x^i} + \frac{\partial \eta_i}{\partial x^j}\right)\,. \tag{28}$$

The strain tensor is clearly symmetric, i.e. $e_{ij} = e_{ji}$. The Lagrangian for small amplitude deformations is

$$L[\eta] = \int_\Omega d^3x \left(\frac{1}{2}\rho \dot{\eta}_i^2 - \frac{1}{2} e_{ij} c^{ijkl} e_{kl}\right)\,. \tag{29}$$

where c^{ijkl} is a rank-4 Cartesian tensor of elastic constants.

a) Show that we may assume that c has symmetries $c_{ijkl} = c_{jikl}$ and $c_{ijkl} = c_{klij}$ following from how it enters L.

SOLUTION: Since i, j, k, l are dummy variables that are summed over, we can always rename them, so

$$e_{ij} c^{ijkl} e_{kl} = e_{ji} c^{jikl} e_{kl} = e_{ij} c^{jikl} e_{kl}\,, \tag{30}$$

where in the last line I've used the fact that e_{ij} is symmetric, and so $e_{ij} = e_{ji}$. Thus we can see that $c_{ijkl} = c_{jikl}$ as required. Similarly, we can rewrite the dummy indices as

$$e_{ij} c^{ijkl} e_{kl} = e_{kl} c^{klij} e_{ij} = e_{ij} c^{klij} e_{kl}\,, \tag{31}$$

where in the last step I simply rearranged the terms. Thus we conclude that $c_{ijkl} = c_{klij}$ as required.

b) Extremize the action, allowing for variation along the boundary of Ω to obtain the equations of motion,

$$\rho \frac{\partial^2 \eta_i}{\partial t^2} - \frac{\partial}{\partial x_j}\sigma_{ji} = 0\,, \tag{32}$$

where $\sigma^{ij} = c^{ijkl} e_{kl}$ is known as the Cartesian stress tensor, and

$$\sigma^{ij}\hat{n}_j = 0\,, \tag{33}$$

where $\hat{n}_j$ are the components of the outward normal on $\partial\Omega$, the boundary of Ω.

SOLUTION:
Let's define

$$\mathcal{L} = \frac{1}{2}\rho\dot{\eta}_i^2 - \frac{1}{2} e_{ij} c^{ijkl} e_{kl}\,. \tag{34}$$

Then extremizing the action means that

$$\int dt \int_\Omega d^3x \left[\frac{\partial \mathcal{L}}{\partial \eta_i}\delta\eta_i + \frac{\partial \mathcal{L}}{\partial(\partial_t \eta_i)}\partial_t(\delta\eta_i) + \frac{\partial \mathcal{L}}{\partial(\partial_j\eta_i)}\partial_j(\delta\eta_i)\right] = 0\,. \tag{35}$$

Integrating the second term by parts in time, and the third term by parts in space, we find

$$\int dt \int_\Omega d^3x \left[\frac{\partial \mathcal{L}}{\partial \eta_i} - \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t\eta_i)} - \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j\eta_i)}\right]\delta\eta_i + \int dt \int_\Omega d^3x\, \partial_j \left[\frac{\partial \mathcal{L}}{\partial(\partial_j\eta_i)}\delta\eta_i\right] = 0\,. \tag{36}$$

The first term corresponds to the usual Euler-Lagrange equation, which we tackle first. We begin by noting that

$$\begin{aligned}\frac{\partial e_{kl}}{\partial(\partial_i\eta_j)} &= \frac{1}{2}\frac{\partial}{\partial(\partial_i\eta_j)}\left(\partial_k\eta_l + \partial_l\eta_k\right)\\ &= \frac{1}{2}\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right)\,.\end{aligned} \tag{37}$$

The Euler-Lagrange equation is then

$$\begin{aligned}\frac{\partial L}{\partial \eta^j} - \partial_i \frac{\partial L}{\partial(\partial_i\eta_j)} = 0 &\implies \partial_t(\rho\dot{\eta}^j) - \frac{1}{2}\partial_i\left[\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right)c^{klmn}e_{mn}\right] = 0\\ &\implies \rho\ddot{\eta}^j - \frac{1}{2}\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right)\partial_i\sigma^{kl} = 0\\ &\implies \rho\ddot{\eta}^j - \frac{1}{2}(\partial_i\sigma^{ij} + \partial_i\sigma^{ji}) = 0\\ &\implies \rho\ddot{\eta}^j - \partial_i\sigma^{ij} = 0\,,\end{aligned} \tag{38}$$

In the second last line, I exploited the symmetry of σ_{ij}, which ultimately stems from the symmetries of c_{ijkl}.
Now for the second term. By the divergence theorem, we can write this as

$$\int dt \int_\Omega d^3x\, \partial_j\left[\frac{\partial\mathcal{L}}{\partial(\partial_j\eta_i)}\delta\eta_i\right] = \int dt \int_{\partial\Omega} dS\, \hat{n}_j \frac{\partial \mathcal{L}}{\partial(\partial_j\eta_i)}\delta\eta_i\,, \tag{39}$$

where $\partial\Omega$ denotes the boundary of Ω, and the integral is over the surface of the boundary with normal vector $\hat{n}_j$ to the surface at all points. Therefore, this term requires that

$$\begin{aligned}\hat{n}_j\frac{\partial\mathcal{L}}{\partial(\partial_j\eta_i)} = 0 &\implies \hat{n}_j\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right)c^{klmn}e_{mn} = 0\\ &\implies \hat{n}_j\sigma^{ij} = 0\,.\end{aligned} \tag{40}$$

c) For an isotropic material, c_{ijkl} must be an isotropic Cartesian tensor, i.e. it must take the same values even if we rotate the coordinates arbitrarily by any 3D rotation. It is a fact that the most general rank-4 isotropic tensor is a linear combination of terms of the form $\delta_{ij}\delta_{kl}$ where the 4 indices can be permuted. Using the symmetries in part *a)*, show that c_{ijkl} must take the form

$$c_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \,, \tag{41}$$

where λ, μ are two material constants, known as the Lamé constants.

SOLUTION: We can write c_{ijkl} as the most general isotropic tensor, going through all possible permutations:

$$c_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu \delta_{ik}\delta_{jl} + \zeta \delta_{il}\delta_{jk} \,. \tag{42}$$

However, we know that

$$c_{ijkl} = c_{jikl} = \lambda \delta_{ij}\delta_{kl} + \zeta \delta_{ik}\delta_{jl} + \mu \delta_{il}\delta_{jk} \implies \mu = \zeta \,, \tag{43}$$

where the last line can be deduced readily by choosing $i = k$ and $j = l$, but $i \neq j$. The other symmetry doesn't help us here, so we conclude that c_{ijkl} must take the form

$$c_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{44}$$

as required.

d) Show that in terms of the Lamé constants, the equations of motion for η reduce to

$$\rho \frac{\partial^2 \eta_i}{\partial t^2} = (\lambda + \mu) \frac{\partial^2 \eta^j}{\partial x^i \partial x^j} + \mu \frac{\partial^2 \eta_i}{\partial x^j \partial x_j} \,. \tag{45}$$

SOLUTION: With this new expression, we find

$$\begin{aligned} \sigma_{ij} &= c_{ijkl} e^{kl} \\ &= [\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})] e^{kl} \\ &= \lambda \delta_{ij} e^k{}_k + 2\mu e_{ij} \\ &= \lambda \delta_{ij} (\partial^k \eta_k) + \mu(\partial_i \eta_j + \partial_j \eta_i) \,. \end{aligned} \tag{46}$$

Thus, the equations of motion are now

$$\begin{aligned} \rho \ddot{\eta}^j - \partial_i \sigma^{ij} = 0 &\implies \rho \ddot{\eta}^j - \lambda \partial^j \partial^k \eta_k - \mu(\partial_i \partial^i \eta^j + \partial_i \partial^j \eta^i) = 0 \\ &\implies \rho \ddot{\eta}^j = (\lambda + \mu) \partial^j \partial^i \eta_i + \mu \partial_i \partial^i \eta^j \,, \end{aligned} \tag{47}$$

as required.

e) By plugging in the plane wave ansatz,

$$\vec{\eta} = \vec{a} e^{i(k_i x^i - \omega t)} \,, \tag{48}$$

deduce that there are two kinds of waves, longitudinal and transverse, and determine their respective phase velocities.

SOLUTION: Plugging in the plane wave ansatz, we find

$$-\omega^2 \rho \eta^j = -(\lambda + \mu) k^j k^i \eta_i - \mu k^2 \eta^j \tag{49}$$

We can decompose η into longitudinal η_L and transverse η_T components, both of which are orthogonal to each other. Furthermore, we have $k^j \eta_{Tj} = 0$ and $k^j \hat{\eta}_{Lj} = k$, where the hat denotes a unit vector in the appropriate direction. Multiplying the entire equation by $\hat{\eta}_{Tj}$ gives

$$(\omega^2 \rho - \mu k^2)\eta_T = 0\,, \tag{50}$$

which is the transverse component wave equation in Fourier space, with a phase velocity of $c_T = \sqrt{\mu/\rho}$. On the other hand, multiplying by $\hat{\eta}_{Lj}$ gives

$$-\omega^2 \rho \eta_L = -(\lambda + \mu) k^2 \eta_L - \mu k^2 \eta_L \implies \left[\omega^2 \rho - (\lambda + 2\mu) k^2\right] \eta_L = 0\,, \tag{51}$$

which is the longitudinal component wave equation in Fourier space, with a phase velocity of $c_L = \sqrt{(\lambda + 2\mu)/\rho}$.