Problem Set 2: Calculus of Variations II

1 SG 1.4: Elastic Rods (20 points)

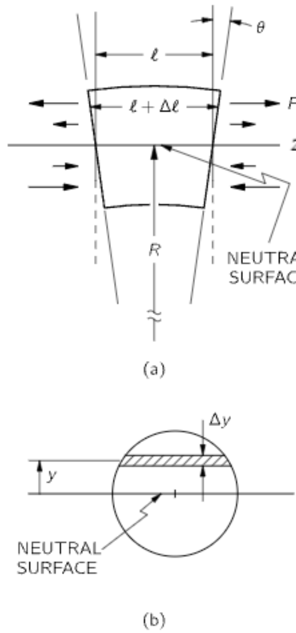

Figure 1: Set up for the elastic rod problem. R is the radius of curvature due to the bending, and z marks the z-axis of the problem. Take the origin to be at the symmetric point of this segment. This figure is taken from the Feynman Lectures on Physics, Chapter 38.

The elastic energy per unit length of a bent steel rod is given by $\frac{1}{2}YI/R^2$. Here, R is the radius of curvature due to the bending, Y is the Young's modulus of the steel, and $I = \iint dxdy\, y^2$ is the moment of inertia of the rod's cross section about an axis through its centroid and perpendicular to the plane in which the rod is bent. You may find Fig. 1 helpful for understanding the set-up.

a) If the rod is only slightly bent into the yz-plane and lies close to the z-axis, show that this elastic energy can be approximated as

$$U[y] = \int_0^L dz\, \frac{1}{2} YI(y'')^2\,, \tag{1}$$

where the prime denotes differentiation with respect to z and L is the length of the rod.

SOLUTION:
I'll write out this solution very carefully since there were many questions about it on Piazza. Consider some segment of the rod at some coordinate z. We have some value of $y(z)$, $y'(z)$ and $y''(z)$. A little bit further on at $z+\Delta z$, these values have shifted to $y(z+\Delta z) = y(z) + \Delta y(z) = y(z) + y'(z)\Delta z$, $y'(z+\Delta z) = y'(z) + \Delta y'(z) = y'(z) + y''(z)\Delta z$.
Referring to Fig. 1, we can see that the angle that is normal to the cross sectional area is defined as $\tan\psi(z) = y'(z)$. But $\psi(z+\Delta z) = \psi(z) + \theta(z)$. Therefore

$$\Delta y'(z) = y''(z)\Delta z = \tan(\psi+\theta) - \tan(\psi) = (1+\tan^2\psi)\theta = (1+y'^2(z))\theta\,. \tag{2}$$

Furthermore, we know that $R\theta = \sqrt{\Delta y^2 + \Delta z^2} = \Delta z\sqrt{1+y'^2}$. Putting everything together, we find

$$y''(z) = \frac{(1+y'^2)^{3/2}}{R}\,. \tag{3}$$

How big could y' be? Well, if $y' \sim 1$ at any point, then you would get large displacements, unless y'' is large too, so that y' can change rapidly. Therefore, as long as we take R to be large enough, we will have $y' \ll 1$. And so, under that assumption, we have

$$y''(z) \simeq \frac{1}{R}\,, \tag{4}$$

and so

$$U[y] = \int_0^L dz\,\frac{1}{2}\frac{YI}{R^2} \simeq \int_0^L dz\,\frac{1}{2}YI(y'')^2\,, \tag{5}$$

as required.
Since $y'' \simeq 1/R$, we also know that that $y' \lesssim L/R$, and therefore $\sqrt{1+y'^2} \simeq 1 + \mathcal{O}(L^2/R^2)$. Therefore any distinction between the original start and end points of the rod along the z-axis and the actual length of the rod is unimportant at $\mathcal{O}(L^2/R^2)$, and so long as we make R big enough, this is completely irrelevant.

b) **Euler's problem: The buckling of a slender column.**— The rod is used as a column which supports a compressive load Mg directed along the z-axis, (which is vertical; see Fig. 2 left).

- Show that when the rod buckles slightly (i.e. deforms with both ends remaining on the z-axis), the total energy, including the gravitational potential energy of the loading mass M, can be approximated by

$$U[y] = \int_0^L dz\,\left[\frac{YI}{2}(y'')^2 - \frac{Mg}{2}(y')^2\right]\,. \tag{6}$$

SOLUTION:
Consider a segment of the rod of length $d\ell = \sqrt{dy^2 + dz^2} = \sqrt{1+y'^2}\,dz$. Suppose the mass is located at height h. Then

$$\int_0^h dz\sqrt{1+y'^2} = L \tag{7}$$

But if the deformation is small, then $y'^2 \ll 1$, and

$$L \simeq \int_0^h dz\,\left(1 + \frac{1}{2}(y')^2\right) = h + \frac{1}{2}\int_0^h dz\,(y')^2 \implies L - h \simeq \frac{1}{2}\int_0^L dz\,(y')^2\,, \tag{8}$$

where the replacement of $h \to L$ in the integral is justified because it introduces a higher order correction to $L-h$. Since the loading mass has potential energy given by $Mg(h-L)$, the total energy is

$$U[y] = \int_0^L dz \left[\frac{YI}{2}(y'')^2 - \frac{Mg}{2}(y')^2\right], \tag{9}$$

as required.

- Obtain a differential equation with suitable boundary conditions that must be obeyed if the potential energy is minimized. In doing this, take care not to throw away any term arising from the integration by parts.

SOLUTION:
Let's make a small deformation $y + \delta y$, with $\delta y(0) = \delta y(L) = 0$. Then the total energy is deformed as

$$\begin{aligned}\delta U &= \int_0^L dz\,[YIy''\delta y'' - Mgy'\delta y'] \\ &= YIy''\delta y'|_0^L - \int_0^L dz\,[YIy''' + Mgy']\,\delta y' \\ &= YIy''\delta y'|_0^L - \int_0^L dz\,[YIy'''' + Mgy'']\,\delta y\,,\end{aligned} \tag{10}$$

where we have simply integrated by parts twice, keeping the boundary terms for the term with $\delta y'$, but discarding the one with δy, since $\delta y(0) = \delta y(L) = 0$. The equation of motion is then obtained by setting $\delta U = 0$, giving

$$YIy'''' + Mgy'' = 0\,, \tag{11}$$

with boundary conditions

$$y''(0) = y''(L) = 0\,. \tag{12}$$

- Show that $y(z) = 0$ is the only allowed solution if $Mg < \pi^2 YI/L^2$. Explain what happens above this threshold.

SOLUTION:
Let's redefine $\eta(z) \equiv y''(z)$, so that the differential equation and boundary conditions become

$$\eta'' + \frac{Mg}{YI}\eta = 0\,, \quad \eta(0) = \eta(L) = 0\,. \tag{13}$$

The solution is like that of the harmonic oscillator,

$$\eta = A\sin\sqrt{\frac{Mg}{YI}}z + B\cos\sqrt{\frac{Mg}{YI}}z\,, \tag{14}$$

and imposing the boundary condition finally gives

$$B = 0\,, \text{and } A = 0 \text{ or } \sqrt{\frac{Mg}{YI}} = \frac{n\pi}{L}\,. \tag{15}$$

for some nonnegative integer n. We can see immediately that if $Mg < \pi^2 YI/L^2$, then

the only solution is $\eta(z) = 0$, giving $y(z) = Cz + D$. However, imposing the boundary conditions $y(0) = y(L) = 0$ gives $y(z) = 0$ as the only solution.
Above this threshold, while $y(z) = 0$ is still a possible solution, other solutions exist, and if the system is perturbed slightly, it may not return to the $y(z) = 0$ solution. Buckling can occur.

c) **Leonardo da Vinci's problem: the light cantilever.**—Here we take the z-axis as horizontal and the y-axis as vertical (see Fig. 2 right). The rod is used as a beam or cantilever and is fixed into a wall so that $y(0) = y'(0) = 0$. A weight Mg is hung from the end $z = L$ and the beam sags in the $(-y)$-direction. We wish to find $y(z)$ for $0 < z < L$. We will ignore the weight of the beam itself.

- Show that the total energy, including the gravitational potential energy of the weight, can be written as

$$U[y] = \int_0^L dz \left[\frac{YI}{2} (y'')^2 + Mgy' \right] . \tag{16}$$

SOLUTION:
The gravitational potential energy of the weight is

$$Mgy = \int_0^L dz \, Mgy' , \tag{17}$$

since $y(0) = 0$. Therefore, the total energy is

$$U[y] = \int_0^L dz \left[\frac{YI}{2} (y'')^2 + Mgy' \right] . \tag{18}$$

as required.

- Find the differential equation and boundary conditions at $z = 0, L$ that arise from minimizing the total energy. In doing this, take care not to throw away any term arising from the integration by parts. You may find the following identity useful:

$$\frac{d}{dz}(f'g'' - fg''') = f''g'' - fg'''' . \tag{19}$$

SOLUTION:
Performing a variation $y \to y + \delta y$, we find

$$\delta U = \int_0^L dz \, [YIy'' \, \delta y'' + Mg \, \delta y'] \, . \tag{20}$$

$$= YIy''\delta y'|_0^L - \int_0^L dz \, [YIy''' - Mg] \, \delta y' \, . \tag{21}$$

At this point we can redefine $\eta \equiv y'$ to find that

$$\delta U = YI\eta'\delta\eta|_0^L - \int_0^L dz \, [YI\eta'' - Mg]\delta\eta \, , \tag{22}$$

which lead to the following differential equation and boundary condition:

$$YI\eta'' - Mg = 0 \, , \quad \eta'(L) = 0 \, . \tag{23}$$

Note that $\delta\eta(0) = 0$ because of the fixed boundary condition $y'(0) = 0$.

- Solve the equation. You should find that the displacement of the end of the beam is $y(L) = -\frac{1}{3}MgL^3/(YI)$.

SOLUTION:
Integrating the differential equation gives

$$\eta(z) = \frac{Mg}{2YI}z^2 + Cz + D \, . \tag{24}$$

Imposing the boundary conditions then leads to

$$\eta(z) = \frac{Mg}{2YI}(z^2 - 2Lz) \tag{25}$$

Given that $\eta = y'$, we can integrate another time to find

$$y(z) = \frac{Mg}{2YI}\left(\frac{z^3}{3} - Lz^2\right) + C \, , \tag{26}$$

and again imposing the boundary condition $y(0) = 0$ gives finally

$$y(z) = \frac{Mg}{6YI}(z - 3L)z^2 \tag{27}$$

Finally, we see that

$$y(L) = -\frac{Mg}{3YI}L^3 \tag{28}$$

as expected.

2 Helical Symmetry (5 points)

Consider a particle of mass m moving in a potential

$$U(\rho, \phi, z) = V(\rho, a\phi + z) \, , \tag{29}$$

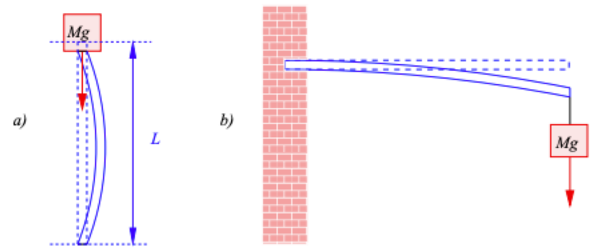

Figure 2: A rod used as (left) a column, and (right) a cantilever. Figure taken from SG.

where (ρ, ϕ, z) are cylindrical coordinates, and a is a constant with dimensions of length. The potential therefore only depends on two variables, even though there are three coordinates.

a) Write down the Lagrangian $L = T - U$ of the particle in cylindrical coordinates.

SOLUTION:
In cylindrical coordinates, the velocity in the ρ, ϕ, and z directions (which are all orthogonal directions) are given respectively by

$$v_\rho = \dot{\rho}\,, \quad v_\phi = \rho\dot{\phi}\,, \quad v_z = \dot{z}\,, \tag{30}$$

where the dots indicate the time derivative. Since we have $v^2 = v_\rho^2 + v_\phi^2 + v_z^2$, the Lagrangian is given by

$$L = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) - V(\rho, a\phi + z)\,. \tag{31}$$

b) Show that the following transformation is a symmetry of the action,

$$\rho \to \rho\,, \quad \phi \to \phi + \zeta\,, \quad z \to z - a\zeta\,. \tag{32}$$

for some arbitrary ζ that is independent of time. Note that ζ is not necessarily small!

SOLUTION:
Under the transformation, $\dot{\rho}$, $\rho\dot{\phi}$ and $\dot{z}$ remain unchanged. Therefore,

$$\begin{aligned} L &\to \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) - V(\rho, a(\phi+\zeta) + (z - a\zeta)) \\ &= \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) - V(\rho, a\phi + z) \qquad (33) \\ &= L\,. \qquad (34) \end{aligned}$$

The Lagrangian is unchanged under the transformation, showing that it is a symmetry of the action.

c) Use Noether's theorem to compute the associated conserved quantity.

SOLUTION:

Noether's theorem tells us that the conserved charge is given by

$$\begin{aligned} Q &= \frac{\partial L}{\partial \dot{\phi}} + \frac{\partial L}{\partial \dot{z}}(-a) \\ &= m\rho^2\dot{\phi} - ma\dot{z} \end{aligned} \tag{35}$$

d) Use the Euler-Lagrange equations to explicitly check that the quantity that you computed in part c) is conserved.

SOLUTION:
For the ϕ coordinate, the Euler-Lagrange equation reads

$$\frac{d}{dt}(m\rho^2\dot{\phi}) - a\frac{\partial V}{\partial \xi} = 0\,, \tag{36}$$

where $\xi \equiv a\phi + z$. For the z coordinate, we have

$$m\ddot{z} - \frac{\partial V}{\partial \xi} = 0\,. \tag{37}$$

Multiplying the last equation by $-a$ and subracting it from the first equation, we find

$$\frac{d}{dt}(m\rho^2\dot{\phi}) - ma\ddot{z} = 0 \implies \frac{d}{dt}(m\rho^2\dot{\phi} - ma\dot{z}) = \frac{dQ}{dt} = 0\,. \tag{38}$$

as required.

e) Argue that the potential created by a single helical strand of DNA (assuming that it is uniformly negatively charged and infinite) has the symmetry in b). A charge moving in this potential thus has the conservation law that you derived.

SOLUTION:
In a helix with axis perpendicular to the horizontal, the coils are tilted an angle α off the horizontal. Suppose we displace a particle by some angle ζ azimuthally. From the perspective of the particle, the helix appears displaced upward (or downward, depending on the handedness of the helix) by an amount $(R\tan\alpha)\zeta$. Therefore, if I also displace the particle by $(R\tan\alpha)\zeta$ (upward or downward), the helix will appear unchanged to it. This is precisely the symmetry transformation in part b).

3 The Laplace-Runge-Lenz Vector (25 points)

Consider the central potential problem, specializing to the potential $V(r) = -\alpha/r$, so that the action is

$$S = \int dt\, \left[\frac{1}{2}m\dot{\vec{r}}^2 + \frac{\alpha}{r}\right]\,. \tag{39}$$

In class, we found two conserved quantities for all central potentials: the angular momentum $\vec{L} = \vec{r}\times\vec{p}$ from the rotational symmetry, and energy E from time-translation invariance. It turns out that for an $1/r$ potential, there is a third conserved quantity, the Laplace-Runge-Lenz vector $\vec{A}$, which is also conserved. This is defined as

$$\vec{A} = \vec{p}\times\vec{L} - m\alpha\hat{r}\,, \tag{40}$$

where $p_i = \partial L/\partial r^i$ is the momentum of the particle, and $\hat{r} = \vec{r}/r$ is the unit vector from the origin to the particle. The symmetry that gives rise to this conserved quantity is not obvious, but is nevertheless present; symmetries that are not manifest are often called "hidden symmetries". The literature on where this symmetry comes from is fascinating, but we won't be concerned with that here.

You may find the identities $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$, and $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$ helpful throughout this problem.

a) Show that the Laplace-Runge-Lenz vector is conserved by explicitly calculating $d\vec{A}/dt$.

SOLUTION:
The time derivative of the Laplace-Runge-Lenz vector is given by

$$\frac{d\vec{A}}{dt} = \frac{d}{dt}\left(\vec{p}\times\vec{L} - m\alpha\hat{r}\right) \tag{41}$$

We'll split the calculation into two parts. First,

$$\frac{d}{dt}(\vec{p}\times\vec{L}) = \frac{d\vec{p}}{dt}\times\vec{L} + \vec{p}\times\frac{d\vec{L}}{dt} = -\frac{\alpha}{r^2}\hat{r}\times\vec{L}\,, \tag{42}$$

where we have made use of the fact that $\vec{L}$ is conserved, and therefore $d\vec{L}/dt = 0$, and that the Newton's second law gives $d\vec{p}/dt = -(\alpha/r^2)\hat{r}$. At this point, we write $\vec{L} = \vec{r}\times\vec{p}$ and simplify the triple vector product, obtaining

$$\begin{aligned}\frac{d}{dt}(\vec{p}\times\vec{L}) &= -\frac{\alpha}{r^2}\hat{r}\times(\vec{r}\times\vec{p}) = -\frac{\alpha}{r^2}\left[(\hat{r}\cdot\vec{p})\vec{r} - (\hat{r}\cdot\vec{r})\vec{p}\right] \\ &= -\frac{\alpha}{r^2}\left[m\frac{dr}{dt}\vec{r} - r\vec{p}\right]\,,\end{aligned} \tag{43}$$

where we have made use of the fact that $\hat{r}\cdot\vec{p}$ just projects out the radial component of the momentum, which is $m(dr/dt)$. On the other hand, we have

$$\begin{aligned}\frac{d}{dt}(m\alpha\hat{r}) &= m\alpha\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) \\ &= m\alpha\left(\frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}\right) \\ &= \frac{\alpha}{r}\vec{p} - \frac{m\alpha}{r^2}\frac{dr}{dt}\vec{r} \\ &= \frac{d}{dt}(\vec{p}\times\vec{L})\,,\end{aligned} \tag{44}$$

from which $d\vec{A}/dt = 0$ is obvious.

b) Prove the following identity:

$$\frac{d\hat{r}^i}{dt} = \frac{1}{r^2}\left(r\dot{r}^i - \frac{r_j\dot{r}^j}{r}r^i\right)\,. \tag{45}$$

SOLUTION:

$$\frac{d\hat{r}^i}{dt} = \frac{d}{dt}\left(\frac{r^i}{r}\right) = \frac{\dot{r}^i}{r} - \frac{1}{r^2}\dot{r}r^i = \frac{1}{r^2}(r\dot{r}^i - \dot{r}r^i) = \frac{1}{r^2}\left(r\dot{r}^i - \frac{r_j\dot{r}^j}{r}r^i\right)\,, \tag{46}$$

c) Consider the transformation $r^i \to r^i + \delta r^i$, where

$$\delta r^i = \varepsilon_j m(2\dot{r}^i r^j - r^i \dot{r}^j - r^k \dot{r}_k \delta^{ij}), \tag{47}$$

where ε_j is an arbitrary vector (with appropriate dimensions), and δ^{ij} is the Kronecker delta. Let's first examine the potential energy term. Show that it transforms as

$$\frac{\alpha}{r} \to \frac{\alpha}{r} + \delta L_1, \quad \delta L_1 = \varepsilon_j \alpha m \frac{d\hat{r}^j}{dt}. \tag{48}$$

SOLUTION:
Let's first examine the variation of r^2. This is

$$\begin{aligned} r^2 \to r^2 + 2r_i \cdot \delta r^i &= r^2 + 2\varepsilon_j m(2r_i \dot{r}^i r^j - r^2 \dot{r}^j - r_i r^k \dot{r}_k \delta^{ij}) \\ &= r^2 + 2\varepsilon_j m(2r_i \dot{r}^i r^j - r^2 \dot{r}^j - r^j r^k \dot{r}_k) \\ &= r^2 + 2\varepsilon_j m(r_i \dot{r}^i r^j - r^2 \dot{r}^j). \end{aligned} \tag{49}$$

This means that

$$\begin{aligned} \frac{1}{r} &\to \frac{1}{\sqrt{r^2 + 2\varepsilon_j m(r_i \dot{r}^i r^j - r^2 \dot{r}^j)}} \\ &= \frac{1}{r}\left[1 - \frac{\varepsilon_j m}{r^2}\left(r_i \dot{r}^i r^j - r^2 \dot{r}^j\right)\right]. \end{aligned} \tag{50}$$

From the identity derived above, we can see that

$$\begin{aligned} \delta L_1 &= -\frac{\alpha \varepsilon_j m}{r^3}\left(r_i \dot{r}^i r^j - r^2 \dot{r}^j\right) \\ &= \varepsilon_j \alpha m \frac{d\hat{r}}{dt}, \end{aligned} \tag{51}$$

d) Next, let's examine the transformation of the kinetic term. Consider the function

$$f^j = m^2\left(\dot{\vec{r}}^2 r^j - (\vec{r} \cdot \dot{\vec{r}})\dot{r}^j\right) \tag{52}$$

Show that the kinetic term transforms as

$$\frac{1}{2}m\dot{\vec{r}}^2 \to \frac{1}{2}m\dot{\vec{r}}^2 + \delta L_2, \quad \delta L_2 = \varepsilon_j \dot{f}^j, \tag{53}$$

and that as a result, the transformation is a symmetry of the action.

SOLUTION:
First, let's take the derivative of f^j. This gives

$$\dot{f}^j = m^2\left(2\dot{r}_i \ddot{r}^i r^j + \dot{\vec{r}}^2 \dot{r}^j - \dot{\vec{r}}^2 \dot{r}^j - r^i \ddot{r}_i \dot{r}^j - (\vec{r} \cdot \dot{\vec{r}})\ddot{r}^j\right) \tag{54}$$

$$= m^2\left(2\dot{r}_i \ddot{r}^i r^j - r^i \ddot{r}_i r^j - (\vec{r} \cdot \dot{\vec{r}})\ddot{r}^j\right) \tag{55}$$

Let's determine the transformation for the velocity, $\dot{r}^i \to \dot{r}^i + \delta \dot{r}^i$. We have

$$\begin{aligned} \delta \dot{r}^i &= \varepsilon_j m(2\ddot{r}^i r^j + 2\dot{r}^i \dot{r}^j - \dot{r}^i \dot{r}^j - r^i \ddot{r}^j - \dot{r}^k \dot{r}_k \delta^{ij} - r^k \ddot{r}_k \delta^{ij}) \\ &= \varepsilon_j m\left[2\ddot{r}^i r^j + \dot{r}^i \dot{r}^j - r^i \ddot{r}^j - (\dot{\vec{r}}^2 + \vec{r} \cdot \ddot{\vec{r}})\delta^{ij}\right] \end{aligned} \tag{56}$$

Then

$$\begin{aligned}\delta L_2 &= m\dot{r}_i \cdot \delta\dot{r}^i \\ &= \varepsilon_j m^2 \left[2\dot{r}_i\ddot{r}^i r^j + \dot{\vec{r}}^2\dot{r}^j - r^i\dot{r}_i\ddot{r}^j - (\dot{\vec{r}}^2 + \vec{r}\cdot\ddot{\vec{r}})\dot{r}^j\right] \\ &= \varepsilon_j m^2 \left[2\dot{r}_i\ddot{r}^i r^j - r^i\dot{r}_i\ddot{r}^j - r^i\ddot{r}_i\dot{r}^j\right] \qquad (57) \\ &= \varepsilon_j \dot{f}^j\,, \qquad (58)\end{aligned}$$

Thus, under the transformation $r^i \to r^i + \delta r^i$, the Lagrangian transforms as

$$L \to L + \delta L_1 + \delta L_2 = L + \epsilon_j \frac{d}{dt}\left(\alpha m\hat{r}^j + f^j\right)\,, \qquad (59)$$

where the last term is a total time derivative. This means that the transformation is a symmetry of the action.

e) Finally, show that the associated Noether charge of this symmetry is the Laplace-Runge-Lenz vector. You may again find the triple vector product identity given at the beginning of the question to be useful.

SOLUTION:
The Noether charge associated with the symmetry is given by

$$Q^j = \alpha m\hat{r}^j + f^j - \frac{\partial L}{\partial\dot{r}^i}\cdot m(2\dot{r}^i r^j - r^i\dot{r}^j - r^k\dot{r}_k\delta^{ij})\,. \qquad (60)$$

where all quantities are to be evaluated on-shell. In that case,

$$\vec{f} = m^2\left(\dot{\vec{r}}^2\vec{r} - (\vec{r}\cdot\dot{\vec{r}})\dot{\vec{r}}\right) = m^2\left[\dot{\vec{r}}\times(\vec{r}\times\dot{\vec{r}})\right] = \vec{p}\times\vec{L}\,, \qquad (61)$$

and

$$\begin{aligned}\frac{\partial L}{\partial\dot{r}^i}\cdot m(2\dot{r}^i r^j - r^i\dot{r}^j - r^k\dot{r}_k\delta^{ij}) &= m^2\dot{r}_i(2\dot{r}^i r^j - r^i\dot{r}^j - r^k\dot{r}_k\delta^{ij}) \\ &= m^2(2\dot{\vec{r}}^2 r^j - r^i\dot{r}_i\dot{r}^j - r^k\dot{r}_k\dot{r}^j) \\ &= 2(\vec{p}\times\vec{L})^j \qquad (62)\end{aligned}$$

Putting everything together, the Noether charge is

$$\vec{Q} = \alpha m\hat{r} + \vec{p}\times\vec{L} - 2(\vec{p}\times\vec{L}) = \alpha m\hat{r} - \vec{p}\times\vec{L}\,, \qquad (63)$$

which is precisely the Laplace-Runge-Lenz vector!

4 SG 1.13: Drums (10 points)

The shape of a (nearly flat) drumskin is described by $h(x, y)$, the height to which the point (x, y) of the flat drumskin is displaced.

a) Show that the area of the drumskin is

$$A[h] = \int_\Omega dx\,dy\,\sqrt{1 + (\partial_x h)^2 + (\partial_y h)^2}\,, \qquad (64)$$

where Ω is the area of the flat (undistorted) drumskin. You may use the fact that parallelogram formed by two vectors $\vec{a}$ and $\vec{b}$ has area $|\vec{a} \times \vec{b}|$.

SOLUTION:
Consider a small patch on the drumskin with one corner of the patch at point (x, y), with area dA. Along the x-direction, the vector bordering this patch is

$$\begin{aligned} d\vec{\ell}_x &= dx\hat{x} + [h(x+dx, y) - h(x, y)]\hat{z} \\ &= dx\,\hat{x} + \partial_x h\,dx\,\hat{z}\,. \end{aligned} \tag{65}$$

Similarly, along the y-direction, the vector bordering this patch is

$$d\vec{\ell}_y = dy\,\hat{y} + \partial_y h\,dy\,\hat{z}\,. \tag{66}$$

The area of the patch is therefore

$$dA = \left|d\vec{\ell}_x \times d\vec{\ell}_y\right| = |-\partial_x h\,dx\,dy\,\hat{x} - \partial_y h\,dx\,dy\,\hat{y} + dx\,dy\,\hat{z}| = dx\,dy\,\sqrt{1 + (\partial_x h)^2 + (\partial_y h)^2}\,, \tag{67}$$

as required.

b) Show that for small distortions, the area reduces to

$$\mathcal{A}[h] = \text{const.} + \frac{1}{2}\int_\Omega dx\,dy\,|\nabla h|^2\,. \tag{68}$$

SOLUTION: With small distortions, you want very small changes to h as we move around the drumskin, i.e. $|\partial_x h|, |\partial_y h| \ll 1$. In this case, we can expand the square root in the area integral to first order in $\partial_x h$ and $\partial_y h$. This gives

$$A[h] \simeq \int_\Omega dx\,dy\,\left[1 + \frac{1}{2}\left((\partial_x h)^2 + (\partial_y h)^2\right)\right] \tag{69}$$

and so we can redefine the area to be

$$\mathcal{A}[h] = \text{const.} + \frac{1}{2}\int_\Omega dx\,dy\,|\nabla h|^2\,, \tag{70}$$

where the constant is just the total flat drumskin area.

c) Show that if h satisfies the two-dimensional Laplace equation $\nabla^2 h = 0$, then the $\mathcal{A}$ is stationary with respect to variations that vanish at the boundary.

SOLUTION:
Under a variation of h, we have

$$|\nabla h|^2 \to |\nabla(h + \delta h)|^2 = (\nabla h)^2 + 2\nabla h \cdot \nabla \delta h \tag{71}$$

the variation of the area is given by

$$\begin{aligned} \delta\mathcal{A} &= \frac{1}{2}\int_\Omega dx\,dy\ [2\nabla h \cdot \nabla\delta h] \\ &= -\int_\Omega dx\,dy\,(\nabla\cdot\nabla h)\delta h\,, \end{aligned} \tag{72}$$

where we have applied the divergence theorem, and dropped the boundary term, since variations on the boundary are not permitted. The area is therefore extremized when

$$\nabla^2 h = 0 \tag{73}$$

as required.

d) Suppose that the flat drumskin has mass ρ_0 per unit area, and surface tension T (the potential energy associated with surface area dA under tension T is $T\,dA$). Write down the Lagrangian controlling the motion of the drumskin, and derive the equation of motion that follows from it.

SOLUTION:
The kinetic energy of a patch is given by

$$dK = \frac{1}{2}\rho_0\,dx\,dy\,\left(\partial_t h\right)^2\,, \tag{74}$$

while the potential energy is $dU = T dA$. Thus, the Lagrangian is

$$L = \int_\Omega dx\,dy\,\mathcal{L} = \int_\Omega dx\,dy\,\left[\frac{1}{2}\rho_0\left(\partial_t h\right)^2 - \frac{1}{2}T|\nabla h|^2\right]\,, \tag{75}$$

based on our arguments from before.
The Euler-Lagrange equation is then

$$\frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial(\partial_\mu h)} = \rho_0\partial_t^2 h - T\nabla^2 h = 0\,, \tag{76}$$

which is the wave equation!