Problem Set 10: Statistics

1 Electromagnetic Discrimination (10 points)

A beam of particles consists of a fraction 10^{-4} electrons and the rest photons. The particles pass through a double-layered detector which gives signals in either zero, one or both layers. The probabilities of these outcomes for electrons (e) and photons (γ) are given by

$$\begin{aligned} P(0|e) &= 0.001\,, \qquad P(0|\gamma) = 0.99899\,, \\ P(1|e) &= 0.01\,, \qquad P(1|\gamma) = 0.001\,, \\ P(2|e) &= 0.989\,, \qquad P(2|\gamma) = 10^{-5}\,. \end{aligned} \tag{1}$$

(a) What is the probability for a particle detected in one layer to be a photon?

SOLUTION:
Using Bayes' theorem,

$$\begin{aligned} P(\gamma|1) &= \frac{P(1|\gamma)P(\gamma)}{P(1)} \\ &= \frac{P(1|\gamma)P(\gamma)}{P(1|e)P(e) + P(1|\gamma)P(\gamma)} \\ &= \frac{0.001(1-10^{-4})}{0.01 \times 10^{-4} + 0.001(1-10^{-4})} \\ &= 0.999\,. \end{aligned} \tag{2}$$

(b) What is the probability for a particle detected in both layers to be an electron?

SOLUTION:
Again, using Bayes' theorem,

$$\begin{aligned} P(e|2) &= \frac{P(2|e)P(e)}{P(2)} \\ &= \frac{P(2|e)P(e)}{P(2|e)P(e) + P(2|\gamma)P(\gamma)} \\ &= \frac{0.989 \times 10^{-4}}{0.989 \times 10^{-4} + 10^{-5}(1-10^{-4})} \\ &= 0.908\,. \end{aligned} \tag{3}$$

2 Spherically Symmetric Distributions (10 points)

Suppose you have 3 independent random variables v_1, v_2, v_3, each following a standard Gaussian distribution, i.e.

$$\phi(v_i) = \frac{1}{\sqrt{2\pi}} e^{-v_i^2/2}\,. \tag{4}$$

Show that the probability density function $h(v)$ of the variable $v = \sqrt{v_1^2 + v_2^2 + v_3^2}$ is the Maxwell-Boltzmann distribution.

SOLUTION:
The probability density function of finding a particle with velocity $\vec{v} = (v_1, v_2, v_3)$ is simply

$$\phi(\vec{v}) = \phi(v_1)\phi(v_2)\phi(v_3)\,, \tag{5}$$

since v_1, v_2, v_3 are independent. Let $\psi(\vec{v})$ be the probability density function in spherical coordinates, i.e.

$$\phi(\vec{v})\, d^3\vec{v} = \psi(\vec{v})\, dr\, d\theta\, d\phi\,. \tag{6}$$

We know from the usual Jacobian going from Cartesian to spherical coordinates that $d^3\vec{v} = v^2 \sin\theta\, dv\, d\theta\, d\phi$. Therefore,

$$\psi(\vec{v}) = \phi(\vec{v}) v^2 \sin\theta\,. \tag{7}$$

The PDF $h(v)$ is then just the marginal PDF of v, integrating out the solid angles. But

$$\phi(\vec{v}) = \frac{1}{(2\pi)^{3/2}} e^{-v^2/2}\,, \tag{8}$$

and has no dependence on angles. We therefore have

$$\begin{aligned} h(v) &= v^2\phi(v) \int d\theta \sin\theta \int d\phi = 4\pi v^2 \phi(v) \\ &= \sqrt{\frac{2}{\pi}} v^2 e^{-v^2/2}\,, \end{aligned} \tag{9}$$

which is a Maxwell-Boltzmann distribution, with velocities normalized to units of $\sqrt{T/m}$.

3 3D Random Walk (50 points)

Consider a particle located at the origin of a 3D Cartesian coordinate system. At each time step, the particle moves a fixed distance a in a random direction. The direction is chosen uniformly at random from all possible directions. All steps are independent from each other. How far away is the particle from the origin after N steps? We would like to derive the probability density function for the distance r of the particle from the origin after N steps.

(a) First, let's consider a single step. Write down the probability density function $f_1(\vec{r})$ (notice the vector sign!) for the particle's position after a single step. Make sure that your PDF is normalized!

SOLUTION:
The particle can move to any point given by a spherical shell around the origin with $|\vec{r}| = a$. The probability density function is therefore

$$f_1(\vec{r}) \propto \delta(r-a)\,, \tag{10}$$

where $r = |\vec{r}|$. But we know that

$$\int d^3\vec{r}\delta(r-a) = \int dr\, 4\pi r^2 \delta(r-a) = 4\pi a^2\,, \tag{11}$$

and therefore

$$f_1(\vec{r}) = \frac{1}{4\pi a^2}\delta(r-a) \tag{12}$$

(b) Let $\vec{R}_N$ be the random variable denoting the position of the particle after N steps. Argue that the probability density function for $\vec{R}_N$, denoted $f_N(\vec{r})$, is given by $f_N = f_1 * f_{N-1}$, where $*$ denotes the convolution operation, and f_{N-1} is the PDF after $N-1$ steps. Therefore, $f_N = f_1 * \cdots * f_1$, N times.

SOLUTION:
We know that $\vec{R}_N = \vec{R}_{N-1} + \vec{R}_1$, i.e. it is the sum of the random variable denoting the position after $N-1$ steps, plus a single displacement. Suppose $\vec{R}_N = \vec{r}_N$. Then, we need $R_{N-1} = r_{N-1}$ and $R_1 = r_1$ such that $\vec{r}_N = \vec{r}_{N-1} + \vec{r}_1$. The probability density function for $\vec{R}_N$ is then obtained by integrating over all possible values of $\vec{r}_{N-1}$ and $\vec{r}_1$: subject to this constraint, weighted by the probability:

$$\begin{aligned} f(\vec{r}_N) &= \int d^3\vec{r}_1 \int d^3\vec{r}_{N-1}\, f_1(\vec{r}_1) f_{N-1}(\vec{r}_{N-1}) \delta^3(\vec{r}_N - (\vec{r}_{N-1} + \vec{r}_1)) \\ &= \int d^3\vec{r}_1\, f_1(\vec{r}_1) f_{N-1}(\vec{r}_N - \vec{r}_1) \\ &= (f_1 * f_{N-1})(\vec{r}_N)\,. \end{aligned} \tag{13}$$

But $f_{N-1} = f_{N-2} * f_1$ etc., and so we have $f_N = f_1 * \cdots * f_1$, N times.

(c) Determine $\tilde{f}_N(\vec{k})$, the Fourier transform of $f_N(\vec{r})$.

SOLUTION:
By the convolution theorem,

$$\tilde{f}_N = \tilde{f}_1^N\,, \tag{14}$$

which is much easier to calculate! The Fourier transform of f_1 is

$$\begin{aligned} \tilde{f}_1(\vec{k}) &= \frac{1}{4\pi a^2} \int d^3\vec{r}\, e^{-i\vec{k}\cdot\vec{r}} \delta(r-a) \\ &= \frac{2\pi}{4\pi a^2} \int_0^\infty dr\, r^2 \delta(r-a) \int_{-1}^{-1} dy\, e^{-ikry}\,, \end{aligned} \tag{15}$$

where $y = \cos\theta$ is the polar angle in spherical coordinates. Continuing the evaluation, we have

$$\begin{aligned} \tilde{f}_1(\vec{k}) &= \frac{1}{2a^2} \int_0^\infty dr\, r^2 \delta(r-a) \int_{-1}^{1} dy\, e^{-ikry} \\ &= \frac{1}{2} \int_{-1}^{1} dy\, e^{-ikay} \\ &= \frac{\sin ka}{ka}\,. \end{aligned} \tag{16}$$

Therefore,

$$\tilde{f}_N(\vec{k}) = \left(\frac{\sin ka}{ka}\right)^N\,. \tag{17}$$

(d) Show that the probability density function $g_N(r)$ for the distance r of the particle from the origin after N steps is given by

$$g_N(r) = \frac{2}{\pi} \int_0^\infty dk\, kr \sin(kr) \left(\frac{\sin ka}{ka}\right)^N\,. \tag{18}$$

Hint: Thinking through the previous question should help you understand the relation between $g_N(r)$ and $f_N(\vec{r})$.

SOLUTION:
First, the PDF $f_N(\vec{r})$ is

$$\begin{aligned} f_N(\vec{r}) &= \int \frac{d^3\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\tilde{f}_N(\vec{k}) \\ &= \int \frac{d^3\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} \left(\frac{\sin ka}{ka}\right)^N . \end{aligned} \tag{19}$$

We can perform this integral in spherical coordinates, which gives

$$\begin{aligned} f_N(\vec{r}) &= \frac{2\pi}{(2\pi)^3}\int_0^\infty dk\, k^2 \left(\frac{\sin ka}{ka}\right)^N \int_{-1}^1 dy\, e^{ikry} \\ &= \frac{1}{2\pi^2}\int_0^\infty dk\, k^2 \left(\frac{\sin ka}{ka}\right)^N \frac{\sin kr}{kr} . \end{aligned} \tag{20}$$

But we also know that

$$g_N(r) = 4\pi r^2 f_N(\vec{r}) , \tag{21}$$

and so finally we have

$$\begin{aligned} g_N(r) &= \frac{2}{\pi}\int_0^\infty dk\, k^2 r^2 \left(\frac{\sin ka}{ka}\right)^N \frac{\sin kr}{kr} \\ &= \frac{2}{\pi}\int_0^\infty dk\, kr\sin(kr) \left(\frac{\sin ka}{ka}\right)^N . \end{aligned} \tag{22}$$

(e) Argue that as $N \to \infty$, the integrand over k in the expression for $g_N(r)$ only has support for $k \ll 1$. By considering $\log[(\sin ka/ka)^N]$, show that

$$\left(\frac{\sin ka}{ka}\right)^N \approx \exp\left[-\frac{N(ka)^2}{6}\right] . \tag{23}$$

SOLUTION:
We note that $[\sin(ka)/(ka)]^N$ goes as $1/(ka)^N$, and so as N goes to infinity, the support of the integral becomes more and more concentrated about the origin. We can therefore perform a Taylor expansion of the function about the origin.
Next,

$$\begin{aligned} \log\left(\frac{\sin ka}{ka}\right)^N &= N\log\left(\frac{\sin ka}{ka}\right) \\ &= N\log\left(1 - \frac{(ka)^2}{3!} + \cdots\right) \\ &= -\frac{N(ka)^2}{6} + \cdots \end{aligned} \tag{24}$$

Thus,

$$\left(\frac{\sin ka}{ka}\right)^N \approx \exp\left[-\frac{N(ka)^2}{6}\right], \tag{25}$$

as required.

(f) Determine $g_N(r)$ for a unit step size, $a = 1$. Show that $\langle r^2 \rangle = N$.

SOLUTION:
At this point,

$$\begin{aligned} g_N(r) &= \frac{2}{\pi}\int_0^\infty dk\, kr \sin(kr) \exp\left[-\frac{N(ka)^2}{6}\right] \\ &= \frac{3\sqrt{6}}{\sqrt{\pi}} \frac{r^2}{N^{3/2}} \exp\left(-\frac{3r^2}{2N}\right). \end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned} \langle r^2 \rangle &= \int_0^\infty dr\, r^2 g_N(r) \\ &= \int_0^\infty dr \frac{3\sqrt{6}}{\sqrt{\pi}} \frac{r^4}{N^{3/2}} \exp\left(-\frac{3r^2}{2N}\right) \\ &= N, \end{aligned} \tag{27}$$

using the magic of Mathematica.

4 Characteristic Functions (15 points)

The **characteristic function** of any random variable X is defined as

$$\phi_X(t) \equiv \langle e^{itX} \rangle. \tag{28}$$

(a) For a continuous random variable defined on $-\infty < x < \infty$ with PDF $f(x)$, show that the characteristic function is related to the Fourier transform $\tilde{f}$ of the PDF by $\phi_X = \tilde{f}^*$.

SOLUTION:
From the definition of the characteristic function,

$$\phi_X(t) = \int_{-\infty}^{\infty} dx\, e^{itx} f(x) = \left[\int_{-\infty}^{\infty} dx\, e^{-itx}\, f(x)\right]^* = \tilde{f}^*(t), \tag{29}$$

as required.

(b) Show that the characteristic function of the exponential distribution with parameter ξ, given by

$$f(x) = \frac{1}{\xi} e^{-x/\xi}, \tag{30}$$

is

$$\phi_X(t) = \frac{1}{1 - it\xi}. \tag{31}$$

SOLUTION:
Following our noses once again,

$$\begin{aligned}\phi_X(t) &= \int_0^\infty dx\, e^{itx} \frac{1}{\xi} e^{-x/\xi} \\ &= \frac{1}{\xi} \int_0^\infty dx\, e^{(it-1/\xi)x} \\ &= \frac{1}{\xi} \left[\frac{e^{(it-1/\xi)x}}{it - 1/\xi} \right]_0^\infty \\ &= \frac{1}{\xi} \frac{1}{1/\xi - it} \\ &= \frac{1}{1 - it\xi} \,,\end{aligned} \tag{32}$$

as required.

(c) By taking derivatives of the characteristic function, show that $\langle x^n \rangle = n! \xi^n$. The lesson to be drawn here is that the characteristic function is extremely efficient at finding moments of a PDF!

SOLUTION:
The key insight is to see that

$$\frac{d^n}{dt^n} \phi_X(t) = \int_0^\infty dx\, (ix)^n e^{itx} \frac{1}{\xi} e^{-x/\xi} \,, \tag{33}$$

and therefore

$$\left. \frac{d^n}{dt^n} \phi_X(t) \right|_{t=0} = i^n \int_0^\infty dx\, x^n \frac{1}{\xi} e^{-x/\xi} = i^n \langle x^n \rangle \,. \tag{34}$$

On the other hand,

$$\frac{d^n}{dt^n} \phi_X(t) = \frac{(-1)^n n! \cdot (-i\xi)^n}{(1 - it\xi)^{n+1}} \implies \left. \frac{d^n}{dt^n} \phi_X(t) \right|_{t=0} = (i\xi)^n n! \tag{35}$$

Comparing the two expressions, we get $\langle x^n \rangle = n! \xi^n$, as required.

5 Call Centers (15 points)

At midday every day, a call center receives calls at a rate (in units of number of calls per hour) given by a random variable M, which follows an exponential distribution,

$$P(M = \mu) = \frac{1}{\xi} e^{-\mu/\xi} \,. \tag{36}$$

On a particular day, given that $M = \mu$, the probability of receiving N calls in an hour is given by a Poisson random variable N, with

$$P(N = n | M = \mu) = \frac{\mu^n}{n!} e^{-\mu} \,. \tag{37}$$

(a) Determine the probability distribution of N, the number of calls received in an hour at midday. Check that it is correctly normalized, i.e. the total probability is 1.

SOLUTION:
Given the conditional probability, we can find

$$\begin{aligned} P(N=n) &= \int_0^\infty d\mu\, P(N=n|M=\mu)P(M=\mu) \\ &= \int_0^\infty d\mu \frac{\mu^n}{n!} e^{-\mu} \frac{1}{\xi} e^{-\mu/\xi} \\ &= \frac{1}{n!\xi} \int_0^\infty d\mu\, \mu^n e^{-\mu(1+1/\xi)} \\ &= \frac{1}{n!\xi} \cdot n! \left(\frac{\xi}{\xi+1} \right)^{n+1} \\ &= \frac{1}{\xi} \left(\frac{\xi}{1+\xi} \right)^{n+1} . \end{aligned} \tag{38}$$

To check that this is correctly normalized, we see that

$$\begin{aligned} \sum_{n=0}^{\infty} P(N=n) &= \frac{1}{\xi} \sum_{n=0}^{\infty} \left(\frac{\xi}{1+\xi} \right)^{n+1} \\ &= \frac{1}{\xi} \frac{\xi}{1+\xi} \frac{1}{1-\xi/(1+\xi)} \\ &= 1 \,, \end{aligned} \tag{39}$$

using the fact that the sum is simply a geometric sum.

(b) Given that the call center received n calls in an hour, find the probability distribution of the rate of calls M, conditioned on this information. Check that this distribution is correctly normalized.

SOLUTION:
We want to find $P(M=\mu|N=n)$ as a function of μ. Using Bayes' theorem, we have

$$\begin{aligned} P(M=\mu|N=n) &= \frac{P(N=n|M=\mu)P(M=\mu)}{P(N=n)} \\ &= \frac{\mu^n}{n!} e^{-\mu} \cdot \frac{1}{\xi} e^{-\mu/\xi} \cdot \left[\frac{1}{\xi} \left(\frac{\xi}{1+\xi} \right)^{n+1} \right]^{-1} \\ &= \frac{\mu^n}{n!} e^{-\mu} e^{-\mu/\xi} \left(\frac{1+\xi}{\xi} \right)^{n+1} . \end{aligned} \tag{40}$$

Once again, to check normalization, we do

$$\begin{aligned} \int_0^\infty d\mu P(M=\mu|N=n) &= \frac{1}{n!} \left(\frac{1+\xi}{\xi} \right)^{n+1} \int_0^\infty d\mu\, \mu^n e^{-\mu(1+1/\xi)} \\ &= \frac{1}{n!} \left(\frac{1+\xi}{\xi} \right)^{n+1} \cdot n! \left(\frac{\xi}{1+\xi} \right)^{n+1} \\ &= 1 \,. \end{aligned} \tag{41}$$

(c) Determine the condition on n such that

$$\mathbb{E}(M|N=n) > \mathbb{E}(M) \,, \tag{42}$$

i.e. the number of phone calls in an hour n the call center must receive such that this new piece of information increases the expected rate of calls per hour above the prior expectation.

SOLUTION:
First,

$$\mathbb{E}(M) = \int_0^\infty d\mu\, \mu \frac{1}{\xi} e^{-\mu/\xi} = \xi\,. \tag{43}$$

But

$$\begin{aligned}\mathbb{E}(M|N=n) &= \int_0^\infty d\mu\, \mu \frac{\mu^n}{n!} e^{-\mu} e^{-\mu/\xi} \left(\frac{1+\xi}{\xi}\right)^{n+1} \\ &= \frac{1+n}{1+\xi}\xi\,.\end{aligned} \tag{44}$$

In order for $\mathbb{E}(M|N=n) > \mathbb{E}(M)$, we need

$$\frac{1+n}{1+\xi}\xi > \xi \implies n > \xi = \mathbb{E}(M)\,, \tag{45}$$

which makes intuitive sense.