Final (Total Time: 1 h 40 min)

Question	**Max. Points**
Q1	40
Q2	20
Q3	20
Q3	60
Total	**140**

1. Do not turn the page until you are instructed to do so.
2. Fill in your name **with your student ID**, and for subject please indicate "PY501 Fall 2024 Final". All other fields can be left blank.
3. **All of your work must be written in the blue exam book.** This document only contains the questions, and will **not** be collected from you.
4. Hand your blue exam booklet in at the end of the exam and before you leave the room.
5. This exam contains **7** pages (including this cover page) and **4** problems.
6. This is a closed book exam. **No books, notes or calculators allowed**.
7. **Some useful facts are provided to you on the two pages after this**. Not all of them may be useful, but you may use them as you see fit.
8. Answers without supporting work will receive no credit.
9. You will be graded based solely on what the graders can understand of your solutions.
10. You are advised to take a quick look at all questions before deciding which ones to tackle first.
11. **Generous partial credit will be given for stating definitions and intermediate steps.**

Useful Information

1. **Complex Functions**. Some definitions for complex functions: given $z = re^{i\theta}$, and a complex number c,

$$\log z = \ln r + i(\theta + 2n\pi)\,, n \in \mathbb{Z}\,, \tag{1}$$
$$\mathrm{Log} z = \ln r + i\Theta\,, -\pi < \Theta \leq \pi\,, \tag{2}$$
$$z^c = e^{c \log z}\,. \tag{3}$$

2. **Poles**. An isolated singular point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}\,, \tag{4}$$

where $\phi(z)$ is analytic and nonzero at z_0. Moreover,

$$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0)\,, (m = 1)\,, \qquad \operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}\,, (m \geq 2)\,, \tag{5}$$

where $\phi^{(m-1)}(z_0)$ is the $(m-1)$-th derivative of $\phi(z)$ evaluated at z_0.

3. **Residue Theorem**. Let C be a closed contour oriented positively. If a function f is analytic inside the contour except for a finite number of singular points z_k, $k = 1, \cdots, n$, inside C, then

$$\int_C dz\, f(z) = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)\,. \tag{6}$$

4. **Fourier Series Basis and Completeness**. Consider a function that is periodic with period L on the entire real line. The Fourier series basis convention we adopt over the domain $-L/2 < x < L/2$ is

$$\phi_n(x) = \frac{1}{\sqrt{L}} e^{i2\pi n x/L}\,, n \in \mathbb{Z}\,, \tag{7}$$

with orthonormality and completeness relations

$$\int_{-L/2}^{L/2} dx\, \phi_m^*(x)\phi_n(x) = \delta_{mn}\,, \qquad \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} \phi_n(x) = \delta(x)\,. \tag{8}$$

5. **Fourier Transform Conventions**. Our conventions for this course are

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x)\,, \qquad f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} \tilde{f}(k)\,. \tag{9}$$

in 1D and

$$\tilde{f}(k) = \int d^3\vec{r}\, e^{-i\vec{k}\cdot\vec{r}} f(\vec{r})\,, \qquad f(\vec{r}) = \int \frac{d^3\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} \tilde{f}(\vec{k})\,, \tag{10}$$

in 3D.

6. **Plancherel's Theorem**. For functions $f(x)$ and $g(x)$, with Fourier transforms $\tilde{f}(k)$ and $\tilde{g}(k)$,

$$\int_{-\infty}^{\infty} dx\, f^*(x) g(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}^*(k)\tilde{g}(k)\,. \tag{11}$$

7. **Convolution**. The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} dy\, f(y) g(x-y)\,. \tag{12}$$

The convolution theorem states that

$$\begin{aligned} \mathcal{F}(f * g)(k) &= [\mathcal{F}f(k)][\mathcal{F}g(k)]\,, \\ \{(fg)(k) &= \frac{1}{2\pi}(\mathcal{F}f * \mathcal{F}g)(k)\,. \end{aligned} \tag{13}$$

8. **Binomial Distribution**. The probability distribution for a binomial random variable X with N trials is

$$P(X = n) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}\,. \tag{14}$$

We also have $\mathbb{E}(X) = Np$ and $\mathrm{Var}(X) = Np(1-p)$.

9. **Poisson Distribution**. The probability distribution for a Poisson random variable X with mean λ is

$$P(X = n) = \frac{\lambda^n e^{-\lambda}}{n!}\,, \tag{15}$$

with $\mathbb{E}(X) = \lambda$ and $\mathrm{Var}(X) = \lambda$.

10. **Gaussian Distribution**. The pdf of a Gaussian random variable with mean μ and variance σ^2 is

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\,. \tag{16}$$

11. **Central Limit Theorem**. This theorem states that the sum n independent, continuous random variables X_i with means μ_i and variances σ_i^2 becomes a Gaussian random variable with mean $\mu = \sum_{i=1}^n \mu_i$ and variance $\sigma^2 = \sum_{i=1}^n \sigma_i^2$ in the limit of $n \to \infty$ under fairly generic assumptions about X_i.

12. **Bayes' Theorem**.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}\,. \tag{17}$$

13. **Geometric Series**. Given the geometric series $S_n = ar^0 + ar^1 + \cdots + ar^n$, the sum is given by

$$S_n = a\left(\frac{1-r^{n+1}}{1-r}\right)\,. \tag{18}$$

You are free to take the $n \to \infty$ limit as long as $|r| < 1$.

1 Another Complex Integral (40 points)

In this problem, we will compute the integral

$$I = \int_0^\infty dx \, \frac{x^{1/4}}{(x+4)(x+2)} \tag{19}$$

by considering an appropriate contour integral of

$$f(z) = \frac{z^{1/4}}{(z+4)(z+2)} \, . \tag{20}$$

Choose a branch cut such that $0 < \arg(z) < 2\pi$. The integration contour is shown in Fig. 1.

(a) *(5 points)* State the location of the branch cut, and the poles of $f(z)$.

> **SOLUTION:**
> The branch cut is along the positive real axis, starting at $z = 0$ and extending to ∞. The poles of $f(z)$ are at $z = -4$ and $z = -2$.

(b) *(5 points)* Obtain the residue of $f(z)$ at each pole.

> **SOLUTION:**
> The residue of $f(z)$ at $z = -4$ is
>
> $$\operatorname*{Res}_{z=-4} f(z) = \frac{(-4)^{1/4}}{(-4+2)} = -\frac{1}{2}(4e^{i\pi})^{1/4} = -\frac{\sqrt{2}e^{i\pi/4}}{2} \, . \tag{21}$$
>
> The residue of $f(z)$ at $z = -2$ is
>
> $$\operatorname*{Res}_{z=-2} f(z) = \frac{(-2)^{1/4}}{(-2+4)} = \frac{1}{2}(2e^{i\pi})^{1/4} = \frac{2^{1/4}e^{i\pi/4}}{2} \, . \tag{22}$$

(c) *(5 points)* Evaluate the integral of $f(z)$ along C_R as $R \to \infty$.

> **SOLUTION:**
> We want to make use of the modulus inequality to show that the integral goes to zero. First, parametrizing $z = Re^{i\theta}$, we have
>
> $$\left| \frac{z^{1/4}}{(z+4)(z+2)} \right| = \frac{R^{1/4}}{|z+4||z+2|} \leq \frac{R^{1/4}}{(R-4)(R-2)} \, . \tag{23}$$
>
> Therefore,
>
> $$\begin{aligned} \lim_{R\to\infty} \int_{C_R} dz \, f(z) &\leq \lim_{R\to\infty} \left(\frac{R^{1/4}}{(R-4)(R-2)} \cdot 2\pi R \right) \\ &= \lim_{R\to\infty} \frac{2\pi R^{5/4}}{R^2} \\ &= 0 \, . \end{aligned} \tag{24}$$

(d) *(5 points)* Evaluate the integral of $f(z)$ along C_ρ as $\rho \to 0$.

SOLUTION:
Once again, we can parametrize the contour by $z = \rho e^{i\theta}$, with

$$\left|\frac{z^{1/4}}{(z+4)(z+2)}\right| = \frac{\rho^{1/4}}{|z+4||z+2|} \leq \frac{\rho^{1/4}}{(\rho-4)(\rho-2)} \,. \tag{25}$$

In the limit of $\rho \to 0$, we have

$$\lim_{\rho\to 0}\left|\frac{z^{1/4}}{(z+4)(z+2)}\right| \leq \lim_{\rho \to 0} \frac{\rho^{1/4}}{8} = 0 \tag{26}$$

and therefore

$$\lim_{\rho\to 0}\int_{C_\rho} dz\, f(z) = 0\,. \tag{27}$$

(e) *(10 points)* Determine the integral of $f(z)$ along the contour C_- in terms of the integral I, in the limit where $R \to \infty$ and $\rho \to 0$.

SOLUTION:
Along C_-, we can use the parametrization $z = re^{i2\pi}$, to write the contour integral as

$$\begin{aligned}\lim_{R\to\infty}\lim_{\rho\to 0}\int_{C_-} dz\, f(z) &= \int_\infty^0 dr \frac{dz}{dr}\frac{(re^{i2\pi})^{1/4}}{(re^{i2\pi}+4)(re^{i2\pi}+2)} \\ &= -\int_0^\infty dr\, \frac{r^{1/4}e^{i\pi/2}}{(r+4)(r+2)} \\ &= -iI\,,\end{aligned} \tag{28}$$

where we note that $e^{i\pi/2} = i$ and $e^{i2\pi} = 1$.

(f) *(10 points)* Evaluate the integral I.

SOLUTION:
The full contour integral is therefore equal to $(1-i)I = \sqrt{2}e^{3\pi i/4}$. By the Cauchy residue theorem, we have

$$\begin{aligned}\sqrt{2}e^{7\pi i/4} I &= 2\pi i\left(-\frac{\sqrt{2}e^{i\pi/4}}{2} + \frac{2^{1/4}e^{i\pi/4}}{2}\right) \\ I &= \pi\cdot e^{-7\pi i/4}e^{i\pi/2}e^{i\pi/4}\left(-1+\frac{1}{2^{1/4}}\right) \\ &= \pi\left(1-\frac{1}{2^{1/4}}\right)\,.\end{aligned} \tag{29}$$

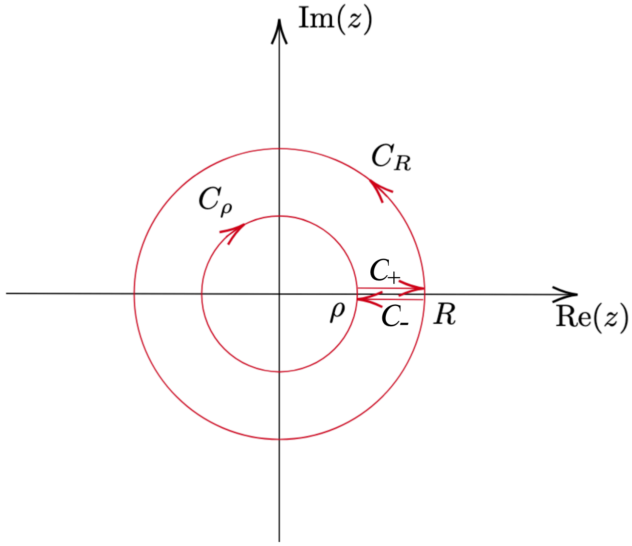

Figure 1: The contour for integrating $f(z)$.

2 Some Properties of the Fourier Transform (20 points)

Let $f(x)$ be a function (possibly complex) with Fourier transform given by $\tilde{f}(k)$.

(a) *(5 points)* Show that the Fourier transform of $f(x+a)$ is $e^{ika}\tilde{f}(k)$, where a is a real constant.

SOLUTION:
The Fourier transform of $f(x+a)$ is

$$\mathcal{F}f(x+a) = \int_{-\infty}^{\infty} dx\, e^{-ikx}\, f(x+a) \tag{30}$$

Performing a change of variables $y = x + a$, we have $dx = dy$, and so the integral becomes

$$\begin{aligned}\mathcal{F}f(x+a) &= \int_{-\infty}^{\infty} dy\, e^{-ik(y-a)}\, f(y) \\ &= e^{ika}\tilde{f}(k)\,,\end{aligned} \tag{31}$$

as required.

(b) *(5 points)* Show that the Fourier transform of $f(ax)$ is $(1/|a|)\tilde{f}(k/a)$, where a is a real constant.

SOLUTION:
The Fourier transform of $f(ax)$ is

$$\mathcal{F}f(ax) = \int_{-\infty}^{\infty} dx\, e^{-ikx}\, f(ax) \tag{32}$$

Performing a change of variables $y = ax$, we have $dy = |a|dx$ if we keep the limits of integration fixed, and so the integral becomes

$$\begin{aligned}\mathcal{F}f(ax) &= \int_{-\infty}^{\infty} \frac{dy}{|a|}\, e^{-iky/a}\, f(y) \\ &= \frac{1}{|a|}\tilde{f}(k/a)\,,\end{aligned} \tag{33}$$

as required.

(c) *(10 points)* Show that if $f(x)$ is real, its Fourier transform $\tilde{f}(k)$ satisfies $\tilde{f}^*(k) = \tilde{f}(-k)$, where * denotes the complex conjugate.

SOLUTION:
The Fourier transform of $f(x)$ is

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x)\,. \tag{34}$$

Taking the complex conjugate of this expression, we have

$$\begin{aligned}\tilde{f}^*(k) &= \int_{-\infty}^{\infty} dx\, e^{ikx} f^*(x) \\ &= \int_{-\infty}^{\infty} dx\, e^{ikx} f(x)\,,\end{aligned} \tag{35}$$

where the last line follows because $f(x)$ is real. But we can see also immediately from the definition that

$$\tilde{f}(-k) = \int_{-\infty}^{\infty} dx\, e^{i-(-k)x} f(x) = \int_{-\infty}^{\infty} dx\, e^{ikx} f(x)\,. \tag{36}$$

Therefore, $\tilde{f}^*(k) = \tilde{f}(-k)$, as required.

3 Smoothing with a Top Hat (20 points)

Let $f(\vec{x})$ be a function in 3D space. A common operation we might want to perform is to smooth a function, i.e. at every point $\vec{x}$, define a new smoothed function $f_s(\vec{x})$ whose value at $\vec{x}$ is given by the average value of $f(\vec{x})$ within a sphere of radius R centered at $\vec{x}$.

(a) *(5 points)* Explain why

$$f_s(\vec{x}) = \frac{3}{4\pi R^3} \int d^3\vec{y}\, f(\vec{y}) \Theta \left(R - |\vec{x} - \vec{y}|\right) \,, \tag{37}$$

where Θ is the Heaviside step function, i.e. $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ otherwise.

SOLUTION:
At point x, we want to average over the value of f in a radius R around x: this corresponds to points $\vec{y}$ such that $|\vec{x} - \vec{y}| \leq R$, which is enforced by the Heaviside step function. The averaging is then performed by integrating over all $f(\vec{y})$, and normalizing by the volume of the sphere, $4\pi R^3/3$.

(b) *(10 points)* Evaluate the 3D Fourier transform of $\Theta(R - r)$, where r is the radial distance from the origin.

(HINT:) You may need to integrate by parts.

SOLUTION:
The Fourier transform of the Heaviside step function is

$$\begin{aligned} \int d^3\vec{r}\, e^{-i\vec{k}\cdot\vec{r}} \Theta(R - r) &= \int_0^\infty dr\, r^2 \Theta(R - r) \cdot 2\pi \int_{-1}^{1} dy\, e^{-ikry} \\ &= 2\pi \int_0^R dr\, r^2 \left[\frac{e^{-ikry}}{-ikr}\right]_{y=-1}^{y=1} \\ &= \frac{4\pi}{k} \int_0^R dr\, r \sin(kr) \,. \end{aligned} \tag{38}$$

Integrating by parts, we find

$$\begin{aligned} \int d^3\vec{r}\, e^{-i\vec{k}\cdot\vec{r}} \Theta(R - r) &= \frac{4\pi}{k} \left[-\frac{r}{k} \cos(kr) \Big|_0^R + \frac{1}{k} \int_0^R dr\, \cos(kr) \right] \\ &= \frac{4\pi}{k} \left[-\frac{R}{k} \cos(kR) + \frac{1}{k^2} \sin(kR) \right] \,. \end{aligned} \tag{39}$$

(c) *(5 points)* Show that

$$\tilde{f}_s(\vec{k}) = \frac{3}{(kR)^3} \left[\sin(kR) - kR\cos(kR)\right] \tilde{f}(\vec{k}) \,, \tag{40}$$

where $\tilde{f}$ and $\tilde{f}_s$ are the Fourier transforms of f and f_s, respectively.

SOLUTION:

Looking at the definition of f_s, we see that

$$f_s = \frac{1}{4\pi R^3}(f * h)\,, \tag{41}$$

where $h(\vec{x}) = \Theta(R - |\vec{x}|)$. Therefore, by the convolution theorem,

$$\tilde{f}_s(\vec{k}) = \frac{1}{4\pi R^3}\tilde{f}(\vec{k})\tilde{h}(\vec{k})\,. \tag{42}$$

But we have already worked out $\tilde{h}(\vec{k})$ in the previous part, and so

$$\begin{aligned}\tilde{f}_s(\vec{k}) &= \frac{3}{4\pi R^3}\cdot\frac{4\pi}{k}\left[-\frac{R}{k}\cos(kR) + \frac{1}{k^2}\sin(kR)\right]\tilde{f}(\vec{k}) \\ &= \frac{3}{(kR)^3}\left[\sin(kR) - kR\cos(kR)\right]\tilde{f}(\vec{k})\,,\end{aligned} \tag{43}$$

as required.

4 Geometric Distribution (60 points)

Consider an experiment where a task with two possible outcomes, success or failure, is repeated until a successful outcome is obtained. This could be e.g. repeatedly flipping a coin until you get heads, or repeatedly throwing a die until you land a 6. Let X be the random variable representing the number of trials needed to achieve success, with at least one trial having to take place.

(HINT:) For this question, you may find the formula provided for a geometric series in the "Useful Information" section helpful.

(a) *(10 points)* Let the probability of success on each trial is p. Explain why the probability distribution is $P(X = n) = (1-p)^{n-1}p$, and show that it is normalized correctly.

SOLUTION:
If $X = n$, then we must have $n-1$ failures followed by a success. The probability of this is $P(X = n) = (1-p)^{n-1}p$.
We see that

$$\sum_{n=1}^{\infty} P(X=n) = \sum_{n=1}^{\infty}(1-p)^{n-1}p = p \cdot \frac{1}{1-(1-p)} = 1\,. \tag{44}$$

(b) *(10 points)* Show that $\mathbb{E}(X) = 1/p$. You may use the fact that

$$\sum_n n x^{n-1} = \frac{d}{dx}\sum_n x^n\,. \tag{45}$$

SOLUTION:
The expectation value is

$$\begin{aligned}\mathbb{E}(X) &= \sum_{n=1}^{\infty} nP(X=n) \\ &= \sum_{n=1}^{\infty} n(1-p)^{n-1}p \\ &= -p\frac{d}{dp}\sum_{n=1}^{\infty}(1-p)^n \\ &= -p\frac{d}{dp}\left[(1-p)\cdot\frac{1}{p}\right] \\ &= \frac{1}{p}\,. \end{aligned} \tag{46}$$

(c) *(5 points)* Given that $\mathbb{E}(X^2) = (2-p)/p^2$, determine $\text{Var}(X)$.

SOLUTION:

We have

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\ &= \frac{2-p}{p^2} - \frac{1}{p^2} \\ &= \frac{1-p}{p^2}\,. \end{aligned} \tag{47}$$

(d) *(5 points)* Show that $P(X \geq m) = (1-p)^{m-1}$.

SOLUTION:
We can evaluate the probability $P(X \geq m)$ as

$$\begin{aligned} P(X \geq m) &= \sum_{n=m}^{\infty} (1-p)^{n-1} p \\ &= p(1-p)^{m-1} \sum_{n=0}^{\infty} (1-p)^n \\ &= p(1-p)^{m-1} \frac{1}{1-(1-p)} \\ &= (1-p)^{m-1}\,. \end{aligned} \tag{48}$$

Alternatively, you can see that $P(X \geq m)$ is just equal to the probability of having $m-1$ failures.

(e) *(5 points)* Determine the probability of needing at least $m+n$ trials to achieve success, for some positive integers m and n, *given* that the first m trials all failed.

SOLUTION:
We want to evaluate the conditional probability $P(X \geq m+n | X \geq m+1)$, which from the definition of conditional probability is

$$\begin{aligned} P(X \geq m+n | X \geq m+1) &= \frac{P([X \geq m+n] \cap [X \geq m+1])}{P(X \geq m+1)} \\ &= \frac{P(X \geq m+n)}{P(X \geq m+1)}\,. \end{aligned} \tag{49}$$

Hence,

$$\begin{aligned} P(X \geq m+n | X \geq m+1) &= \frac{P(X \geq m+n)}{P(X \geq m+1)} \\ &= \frac{(1-p)^{m+n-1}}{(1-p)^m} \\ &= (1-p)^{n-1}\,. \end{aligned} \tag{50}$$

This is the same as $P(X \geq n)$, and therefore the geometric distribution also satisfies the memoryless property of the exponential distribution. The exponential distribution can, in fact, be viewed as the continuous limit of a geometric distribution.

(f) *(15 points)* Suppose you ran the same experiment N times (e.g. a single experiment may be flipping a coin until you got heads; repeat this experiment N times with the same coin), obtaining N values for the number of trials needed to achieve success, $x_1, \cdots, x_N$. Assuming that all the experiments are independent, determine $\hat{p}$, the maximum likelihood estimate for p.

SOLUTION:
Let the probability of obtaining x_i in each experiment, assuming that the probability of success on each trial is p, be $P(x_i|p)$. Since each experiment is independent, The likelihood function is

$$\begin{aligned} L(p) &= P(x_1|p)P(x_2|p)\cdots P(x_N|p) \\ &= (1-p)^{x_1-1}p(1-p)^{x_2-1}p\cdots(1-p)^{x_N-1}p \\ &= p^N(1-p)^{x_1+x_2+\cdots+x_N-N}\,. \end{aligned} \tag{51}$$

The log-likelihood function is

$$\log L(p) = N\log p + \left(\sum_{i=1}^N x_i - N\right)\log(1-p)\,. \tag{52}$$

Taking the derivative with respect to p, we find

$$\begin{aligned} \frac{d}{dp}\log L(p) &= \frac{N}{p} - \frac{\sum_{i=1}^N x_i - N}{1-p} \\ &= \frac{N - p\sum_{i=1}^N x_i}{p(1-p)}\,. \end{aligned} \tag{53}$$

The maximum likelihood estimator $\hat{p}$ is therefore given by

$$\frac{N - \hat{p}\sum_{i=1}^N x_i}{\hat{p}(1-\hat{p})} = 0 \implies \hat{p} = \frac{N}{\sum_{i=1}^N x_i}\,. \tag{54}$$

(g) *(10 points)* We now want to test if a coin is fair by performing just one experiment: flipping the coin until we obtain heads. The result of the experiment is 5 tails, followed by a successful heads, for a total of 6 trials. Construct a one-tailed hypothesis test to determine if the coin is fair. Do this by: i) stating your null hypothesis clearly, ii) computing the p-value for this experiment, and iii) defining what the p-value means.

SOLUTION:
The null hypothesis is that the coin is fair, i.e. $p = 1/2$. The p-value is the probability of obtaining 5 or more tails followed by a succesful heads, which is

$$p = (1-0.5)^{6-1} = \frac{1}{32}\,. \tag{55}$$

The p-value is the probability of obtaining 5 or more tails followed by a successful heads, assuming that the coin is fair.