Problem Set 9: Radiation

1 Magnetic Dipole (10 points)

In class, we discussed the magnetic dipole somewhat briefly. In this problem, you will derive some of the main results on your own. Feel free to follow the discussion in Griffiths to help you along.

Consider a wire loop of radius b, around which we drive an alternating current

$$I(t) = I_0 \cos \omega t \,, \tag{1}$$

resulting in an oscillating magnetic dipole

$$\mathbf{m}(t) = \pi b^2 I(t) \hat{\mathbf{z}} \,. \tag{2}$$

(a) State the perfect dipole and far-field approximations.

> To simplify the radiation fields, we assume the source is small and the observer is far away. The perfect dipole (small source) approximation assumes the radius b is much smaller than the distance r and the wavelength λ:
>
> $$b \ll r, \quad kb \ll 1 \,. \tag{3}$$
>
> The far-field (radiation zone) approximation focuses on the region where the distance from the origin is much larger than the wavelength:
>
> $$kr \gg 1 \,. \tag{4}$$
>
> In this regime, we drop terms falling off as $1/r^2$ or $1/r^3$ in favor of the $1/r$ radiation terms.

(b) Determine $V(r, \theta, t)$ and $\mathbf{A}(r, \theta, t)$ under these approximations, where r is the distance from the origin, θ is the polar angle with respect to the z-axis, and t is the time.

> Since the loop is electrically neutral, the scalar potential is:
>
> $$V(r, \theta, t) = 0 \,. \tag{5}$$
>
> For the vector potential, we start from the retarded vector potential for a current loop:
>
> $$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \oint \frac{I(t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}\mathbf{l}' \,. \tag{6}$$
>
> For a loop in the xy-plane, $\mathbf{r}' = b(\cos \phi' \hat{\mathbf{x}} + \sin \phi' \hat{\mathbf{y}})$ and $\mathrm{d}\mathbf{l}' = b(-\sin \phi' \hat{\mathbf{x}} + \cos \phi' \hat{\mathbf{y}}) \mathrm{d}\phi'$. We evaluate at a point $\mathbf{r}$ in the xz-plane ($\phi = 0$):
>
> $$|\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 + b^2 - 2rb \sin \theta \cos \phi'} \simeq r \left(1 - \frac{b}{r} \sin \theta \cos \phi' \right) \tag{7}$$
>
> The current term becomes:
>
> $$I(t - |\mathbf{r} - \mathbf{r}'|/c) \simeq I_0 \cos \left[\omega \left(t - \frac{r}{c} + \frac{b}{c} \sin \theta \cos \phi' \right) \right] \tag{8}$$
>
> Expanding this using $\cos(x + y) = \cos x \cos y - \sin x \sin y$, and noting that $\omega b/c \ll 1$ (perfect dipole):
>
> $$I \simeq I_0 \left[\cos \omega t_0 - \left(\frac{\omega b}{c} \sin \theta \cos \phi' \right) \sin \omega t_0 \right] \tag{9}$$

where $t_0 = t - r/c$. Substituting this and $1/|\mathbf{r} - \mathbf{r}'| \simeq (1/r)(1 + \frac{b}{r}\sin\theta\cos\phi')$ into the integral:

$$\mathbf{A} \simeq \frac{\mu_0 I_0 b}{4\pi r}\int_0^{2\pi}\left(\cos\omega t_0 + \frac{b\sin\theta\cos\phi'}{r}\cos\omega t_0 - \frac{\omega b\sin\theta\cos\phi'}{c}\sin\omega t_0\right)\cos\phi' \mathrm{d}\phi' \hat{\mathbf{y}} \tag{10}$$

The $\cos\omega t_0$ term integrates to zero. The remaining terms involve $\int_0^{2\pi}\cos^2\phi' \mathrm{d}\phi' = \pi$:

$$\mathbf{A}(r,\theta,t) = \frac{\mu_0 m_0 \sin\theta}{4\pi}\left[\frac{\cos\omega t_0}{r^2} - \frac{\omega\sin\omega t_0}{rc}\right]\hat{\boldsymbol{\phi}} \tag{11}$$

In the far-field radiation zone ($r \gg c/\omega$):

$$\mathbf{A}(r,\theta,t) = -\frac{\mu_0 m_0 \omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin[\omega(t - r/c)]\hat{\boldsymbol{\phi}} \tag{12}$$

(c) Determine $\mathbf{E}(r,\theta,t)$ and $\mathbf{B}(r,\theta,t)$.

The electric field is $\mathbf{E} = -\partial_t \mathbf{A}$:

$$\mathbf{E}(r,\theta,t) = -\frac{\partial}{\partial t}\left[-\frac{\mu_0 m_0\omega}{4\pi c}\frac{\sin\theta}{r}\sin[\omega(t - r/c)]\right]\hat{\boldsymbol{\phi}} \tag{13}$$

$$= \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\hat{\boldsymbol{\phi}} \tag{14}$$

The magnetic field in the radiation zone is $\mathbf{B} \simeq \frac{1}{c}(\hat{\mathbf{r}} \times \mathbf{E})$:

$$\mathbf{B}(r,\theta,t) = \frac{1}{c}\left[\hat{\mathbf{r}} \times \left(\frac{\mu_0 m_0 \omega^2 \sin\theta}{4\pi c r}\cos[\omega t_r]\right)\hat{\boldsymbol{\phi}}\right] \tag{15}$$

$$= -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\hat{\boldsymbol{\theta}} \tag{16}$$

(d) Determine $\mathbf{S}(r,\theta,t)$ and the time-averaged total radiated power $\langle P\rangle$.

The Poynting vector is $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$:

$$\mathbf{S} = \frac{1}{\mu_0}(E_\phi \hat{\boldsymbol{\phi}} \times B_\theta \hat{\boldsymbol{\theta}}) = -\frac{E_\phi B_\theta}{\mu_0}\hat{\mathbf{r}} \tag{17}$$

$$= \frac{\mu_0 m_0^2 \omega^4}{16\pi^2 c^3}\frac{\sin^2\theta}{r^2}\cos^2[\omega(t - r/c)]\hat{\mathbf{r}} \tag{18}$$

Time-averaging over one cycle ($\langle\cos^2\rangle = 1/2$):

$$\langle\mathbf{S}\rangle = \frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}} \tag{19}$$

Integrating over a sphere to find the total power:

$$\langle P\rangle = \int\langle\mathbf{S}\rangle\cdot\mathrm{d}\mathbf{a} = \frac{\mu_0 m_0^2\omega^4}{32\pi^2 c^3}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\frac{\sin^2\theta}{r^2}(r^2\sin\theta\mathrm{d}\theta) \tag{20}$$

$$= \frac{\mu_0 m_0^2\omega^4}{16\pi c^3}\int_0^{\pi}\sin^3\theta\,\mathrm{d}\theta = \frac{\mu_0 m_0^2\omega^4}{16\pi c^3}\left(\frac{4}{3}\right) \tag{21}$$

$$= \frac{\mu_0 m_0^2\omega^4}{12\pi c^3} \tag{22}$$

2 The Multipole Expansion of Radiation (10 points)

In class, we considered a charge and current distribution as a function of $\boldsymbol{r}'$ confined to a region of size R, and wanted to determine the electric and magnetic fields at a position $\boldsymbol{r}$, with $r' < R \ll r$. We derived expressions for the scalar and vector potential for an arbitrary charge distribution in the perfect dipole approximation, given by

$$\begin{aligned} V(\boldsymbol{r}, t) &\simeq \frac{1}{4\pi\epsilon_0 r}\left[Q + \frac{\hat{\boldsymbol{r}}}{r}\cdot\mathbf{p}(t_0) + \hat{\boldsymbol{r}}\cdot\dot{\mathbf{p}}(t_0)\right], \\ \mathbf{A}(\boldsymbol{r}, t) &\simeq \frac{\mu_0}{4\pi}\frac{\dot{\boldsymbol{p}}(t_0)}{r}, \end{aligned} \tag{23}$$

where $t_0 = t - r$, and the total charge Q and dipole $\mathbf{p}$ are defined as

$$Q = \int \mathrm{d}^3\boldsymbol{r}'\,\rho(\boldsymbol{r}'), \qquad \mathbf{p}(t_0) = \int \mathrm{d}^3\boldsymbol{r}'\,\boldsymbol{r}'\rho(\boldsymbol{r}', t_0). \tag{24}$$

Once again, you should feel free to follow the discussion in Griffiths to complete this question.

(a) Compute $\mathbf{E}$ and $\mathbf{B}$ in terms of $\boldsymbol{r}$ and $\boldsymbol{p}$ (and time derivatives as necessary), taking the far-field approximation, i.e. discarding all $1/r^2$ terms in these expressions.

> We start from the potentials in the far-field approximation. For $Q = 0$, the radiation terms (dropping $1/r^2$) are:
>
> $$V(\mathbf{r}, t) \simeq \frac{1}{4\pi\epsilon_0 r}[\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)], \qquad \mathbf{A}(\mathbf{r}, t) \simeq \frac{\mu_0}{4\pi r}\dot{\mathbf{p}}(t_0) \tag{25}$$
>
> We find the fields using $\mathbf{E} = -\nabla V - \partial_t\mathbf{A}$ and $\mathbf{B} = \nabla\times\mathbf{A}$. For radiation, we assume $Q = 0$ and discard terms falling off faster than $1/r$.
> First, the gradient of the scalar potential. Note that $\nabla t_0 = -\nabla r = -\hat{\mathbf{r}}$. In the far-field:
>
> $$\begin{aligned} \nabla V &\simeq \frac{1}{4\pi\epsilon_0 r}\nabla(\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)) \simeq \frac{1}{4\pi\epsilon_0 r}\left[\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0)\right]\nabla t_0 \qquad (26)\\ &= -\frac{1}{4\pi\epsilon_0 r}(\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0))\hat{\mathbf{r}} \qquad (27) \end{aligned}$$
>
> The time derivative of the vector potential is:
>
> $$\frac{\partial\mathbf{A}}{\partial t} = \frac{\mu_0}{4\pi r}\ddot{\mathbf{p}}(t_0) \tag{28}$$
>
> Plugging this into the expression for $\mathbf{E}$ and using $\mu_0 = 1/\epsilon_0$, we get
>
> $$\mathbf{E} = \frac{\mu_0}{4\pi r}\left[(\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}})\hat{\mathbf{r}} - \ddot{\mathbf{p}}\right] = \frac{\mu_0}{4\pi r}\left[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\ddot{\mathbf{p}}(t_0))\right], \tag{29}$$
>
> For the magnetic field, we use the curl of $\mathbf{A}$:
>
> $$\begin{aligned} \mathbf{B} = \nabla\times\mathbf{A} &\simeq \frac{\mu_0}{4\pi r}(\nabla t_0\times\ddot{\mathbf{p}}) \qquad (30)\\ &= -\frac{\mu_0}{4\pi r}(\hat{\mathbf{r}}\times\ddot{\mathbf{p}}(t_0)) \qquad (31) \end{aligned}$$

(b) Choose a spherical coordinates such that the z-axis is in the direction of $\ddot{\mathbf{p}}(t_0)$. Determine the Poynting vector $\mathbf{S}(\boldsymbol{r}, t)$, and the total radiated power.

Let $\ddot{\mathbf{p}} = \ddot{p}\hat{\mathbf{z}}$. In spherical coordinates, $\hat{\mathbf{r}} \times \hat{\mathbf{z}} = -\sin\theta\hat{\boldsymbol{\phi}}$. The fields become:

$$\mathbf{B} = \frac{\mu_0 \ddot{p}(t_0)\sin\theta}{4\pi r}\hat{\boldsymbol{\phi}} \tag{32}$$

$$\mathbf{E} = \mathbf{B} \times \hat{\mathbf{r}} = \frac{\mu_0 \ddot{p}(t_0)\sin\theta}{4\pi r}\hat{\boldsymbol{\theta}} \tag{33}$$

The Poynting vector is $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$:

$$\mathbf{S} = \frac{1}{\mu_0}\left(\frac{\mu_0 \ddot{p}(t_0)\sin\theta}{4\pi r}\right)^2 (\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}}) \tag{34}$$

$$= \frac{\mu_0[\ddot{p}(t_0)]^2\sin^2\theta}{16\pi^2 r^2}\hat{\mathbf{r}} \tag{35}$$

To find the total power, we integrate over a sphere:

$$P = \oint \mathbf{S}\cdot \mathrm{d}\mathbf{a} = \int_0^{2\pi}\mathrm{d}\phi \int_0^{\pi}\left(\frac{\mu_0[\ddot{p}(t_0)]^2\sin^2\theta}{16\pi^2 r^2}\right) r^2\sin\theta \mathrm{d}\theta \tag{36}$$

$$= \frac{\mu_0[\ddot{p}(t_0)]^2}{8\pi}\int_0^{\pi}\sin^3\theta\mathrm{d}\theta = \frac{\mu_0[\ddot{p}(t_0)]^2}{8\pi}\left(\frac{4}{3}\right) \tag{37}$$

$$= \frac{\mu_0[\ddot{p}(t_0)]^2}{6\pi} \tag{38}$$

(c) Consider a point charge q with position $d(t)$. Show that the power radiated by the point charge is

$$P = \frac{\mu_0 q^2 a^2}{6\pi}\,, \tag{39}$$

where a is the acceleration of the charge. This formula is known as the **Larmor formula**.

For a point charge q at position $\mathbf{d}(t)$, the electric dipole moment is defined as:

$$\mathbf{p}(t) = q\mathbf{d}(t) \tag{40}$$

Taking the second time derivative, we get:

$$\ddot{\mathbf{p}}(t) = q\ddot{\mathbf{d}}(t) = q\mathbf{a} \tag{41}$$

where $\mathbf{a}$ is the acceleration of the charge. Substituting this into the total power expression derived in part (b):

$$P = \frac{\mu_0(qa)^2}{6\pi} = \frac{\mu_0 q^2 a^2}{6\pi} \tag{42}$$

3 Radiation Resistance (10 points)

Consider the electric dipole of length d. We define the **radiation resistance** of the electric dipole as the resistance that would give the same average power loss to heat as the oscillating dipole in fact puts out in the form of radiation.

(a) Show that

$$R = 790\left(\frac{d}{\lambda}\right)^2\Omega\,, \tag{43}$$

where λ is the wavelength of the emitted radiation.

Consider an oscillating electric dipole $\mathbf{p}(t) = p_0 \cos(\omega t)\hat{\mathbf{z}}$. For a physical dipole of length d with current $I(t) = I_0 \cos(\omega t)$, the charge is $q(t) = \int I(t)\mathrm{d}t = (I_0/\omega)\sin(\omega t)$. The dipole moment magnitude is:

$$p(t) = q(t)d = \frac{I_0 d}{\omega}\sin(\omega t) \tag{44}$$

$$\ddot{p}(t) = -I_0 d\omega \sin(\omega t) \tag{45}$$

The time-averaged power radiated by an electric dipole (re-inserting c into the result from the previous problem to get SI units) is:

$$\langle P \rangle = \frac{\mu_0 \langle [\ddot{p}(t_0)]^2 \rangle}{6\pi c} = \frac{\mu_0 (I_0 d\omega)^2}{12\pi c} \tag{46}$$

The average power dissipated by a resistance R is $\langle P \rangle = \frac{1}{2} I_0^2 R$. Equating the two:

$$\frac{1}{2} I_0^2 R = \frac{\mu_0 I_0^2 d^2 \omega^2}{12\pi c} \implies R = \frac{\mu_0 d^2 \omega^2}{6\pi c} \tag{47}$$

Substituting $\omega = 2\pi c/\lambda$ and $\mu_0 c = \sqrt{\mu_0/\epsilon_0} \simeq 120\pi\,\Omega$:

$$R = \frac{\mu_0 c^2 (2\pi/\lambda)^2 d^2}{6\pi c} = \frac{2\pi\mu_0 c}{3}\left(\frac{d}{\lambda}\right)^2 \tag{48}$$

$$\simeq \frac{2\pi(120\pi)}{3}\left(\frac{d}{\lambda}\right)^2 = 80\pi^2 \left(\frac{d}{\lambda}\right)^2 \tag{49}$$

$$\simeq 790 \left(\frac{d}{\lambda}\right)^2 \Omega \tag{50}$$

(b) Suppose you have an ordinary radio with a dipole antenna (say $d = 5\,\mathrm{cm}$), should you worry about the radiative contribution of a dipole antenna to the total resistance?

The wavelength for standard FM radio ($\sim$ 100 MHz) is $\lambda = c/f \simeq 3$ m. Using FM as an example:

$$R \simeq 790 \left(\frac{0.05}{3}\right)^2 \simeq 0.22\,\Omega \tag{51}$$

This resistance is very small compared to the typical ohmic resistance of the wires or the internal impedance of the receiver circuitry. Unless the antenna is tuned (where $d \simeq \lambda/2$), the radiative resistance is generally negligible for receiving antennas.

(c) What is the radiation resistance for the oscillating magnetic dipole of size d? Express your answer in terms of λ and d.

For a magnetic dipole loop of area $A = \pi(d/2)^2$, the moment is $m(t) = I(t)A$. Using the power formula derived previously (in SI units):

$$\langle P \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3} = \frac{\mu_0 (I_0 A)^2 \omega^4}{12\pi c^3} \tag{52}$$

Setting $\langle P\rangle = \frac{1}{2}I_0^2 R_m$:

$$\begin{aligned} R_m &= \frac{\mu_0 A^2\omega^4}{6\pi c^3} = \frac{\mu_0(\pi d^2/4)^2(2\pi c/\lambda)^4}{6\pi c^3} && (53)\\ &= \frac{\mu_0 c\pi^5}{6}\left(\frac{d}{\lambda}\right)^4 && (54)\\ &\simeq \frac{(120\pi)\pi^5}{6}\left(\frac{d}{\lambda}\right)^4 && (55)\\ &\simeq 19200\left(\frac{d}{\lambda}\right)^4\,\Omega && (56)\end{aligned}$$

Magnetic dipole radiation is suppressed by an additional factor of $(d/\lambda)^2$ relative to electric dipole radiation.

4 Radiation from a Surface Current (15 points)

Suppose the (electrically neutral) yz-plane carries a time-dependent but uniform surface current $K(t)\hat{\mathbf{z}}$.

(a) Find the electric and magnetic fields at a height x above the plane if a constant current is turned on at $t = 0$, i.e.

$$K(t) = \begin{cases} 0\,, & t \leq 0\,,\\ K_0\,, & t > 0\,. \end{cases} \tag{57}$$

Since the plane is electrically neutral, $V = 0$. The vector potential for the surface current is

$$\mathbf{A}(x,t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(t - |\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}a'\,. \tag{58}$$

Using polar coordinates on the plane (s', ϕ'), where $|\mathbf{r}-\mathbf{r}'| = \sqrt{x^2+s'^2}$ and $\mathrm{d}a' = s'\mathrm{d}s'\mathrm{d}\phi'$:

$$\begin{aligned} \mathbf{A}(x,t) &= \frac{\mu_0\hat{\mathbf{z}}}{4\pi}\int_0^{2\pi}\mathrm{d}\phi'\int_0^\infty \frac{K(t-\sqrt{x^2+s'^2})}{\sqrt{x^2+s'^2}}s'\mathrm{d}s' && (59)\\ &= \frac{\mu_0\hat{\mathbf{z}}}{2}\int_x^\infty K(t-s)\mathrm{d}s && (60)\end{aligned}$$

where we used the substitution $s = \sqrt{x^2+s'^2}$. For a step function K_0, the integrand is non-zero only if $t - s > 0 \implies s < t$. If $t < x$, the field hasn't arrived and $\mathbf{A} = 0$. For $t > x$:

$$\mathbf{A}(x,t) = \frac{\mu_0 K_0\hat{\mathbf{z}}}{2}\int_x^t \mathrm{d}s = \frac{\mu_0 K_0}{2}(t-x)\hat{\mathbf{z}} \tag{61}$$

The fields (for $t > x$) are:

$$\begin{aligned} \mathbf{E} &= -\partial_t\mathbf{A} = -\frac{\mu_0 K_0}{2}\hat{\mathbf{z}} && (62)\\ \mathbf{B} &= \nabla\times\mathbf{A} = -\partial_x A_z\hat{\mathbf{y}} = \frac{\mu_0 K_0}{2}\hat{\mathbf{y}} && (63)\end{aligned}$$

(b) Find the electric and magnetic fields at a height x above the plane if a linearly increasing current is turned on at $t = 0$, i.e.

$$K(t) = \begin{cases} 0\,, & t \leq 0\,, \\ \alpha t\,, & t > 0\,. \end{cases} \tag{64}$$

Using the integral derived above for $t > x$:

$$\begin{aligned}
\mathbf{A}(x,t) &= \frac{\mu_0 \hat{\mathbf{z}}}{2} \int_x^t \alpha(t-s)\mathrm{d}s && (65) \\
&= \frac{\mu_0 \alpha \hat{\mathbf{z}}}{2} \left[ts - \frac{1}{2}s^2 \right]_x^t = \frac{\mu_0 \alpha \hat{\mathbf{z}}}{2} \left[t^2 - \frac{1}{2}t^2 - (tx - \frac{1}{2}x^2) \right] && (66) \\
&= \frac{\mu_0 \alpha}{4}(t-x)^2 \hat{\mathbf{z}} && (67)
\end{aligned}$$

The fields for $t > x$ are:

$$\mathbf{E} = -\partial_t \mathbf{A} = -\frac{\mu_0 \alpha}{2}(t-x)\hat{\mathbf{z}} \tag{68}$$

$$\mathbf{B} = -\partial_x A_z \hat{\mathbf{y}} = \frac{\mu_0 \alpha}{2}(t-x)\hat{\mathbf{y}} \tag{69}$$

(c) Show that for arbitrary $K(t)$, the retarded vector potential can be written in the form

$$\mathbf{A}(x,t) = \frac{\mu_0}{2}\hat{\mathbf{z}} \int_0^\infty \mathrm{d}u\, K\left(t - x - u\right)\,. \tag{70}$$

From part (a), we had

$$\mathbf{A}(x,t) = \frac{\mu_0 \hat{\mathbf{z}}}{2} \int_x^\infty K(t-s)\mathrm{d}s\,. \tag{71}$$

Perform a change of variables $u = s - x$:

$$\begin{aligned}
\mathbf{A}(x,t) &= \frac{\mu_0 \hat{\mathbf{z}}}{2} \int_0^\infty K(t-(u+x))\mathrm{d}u && (72) \\
&= \frac{\mu_0}{2}\hat{\mathbf{z}} \int_0^\infty \mathrm{d}u\, K(t-x-u) && (73)
\end{aligned}$$

(d) Show that the total power radiated per unit area of surface is

$$P(t) = \frac{\mu_0}{2}[K(t_0)]^2\,, \tag{74}$$

where $t_0 = t - x$.

First, we find the fields for an arbitrary $K(t)$ using the form for $\mathbf{A}$ from part (c) and noting that $\frac{\partial K}{\partial t} = K'$:

$$\mathbf{E} = -\partial_t \mathbf{A} = -\frac{\mu_0 \hat{\mathbf{z}}}{2}\int_0^\infty \frac{\partial}{\partial t} K(t_0 - u)\mathrm{d}u \tag{75}$$

$$= -\frac{\mu_0 \hat{\mathbf{z}}}{2}\int_0^\infty K'(t_0 - u)\mathrm{d}u \tag{76}$$

$$= -\frac{\mu_0 \hat{\mathbf{z}}}{2}[-K(t_0 - u)]_0^\infty \tag{77}$$

$$= -\frac{\mu_0 K(t_0)}{2}\hat{\mathbf{z}} \tag{78}$$

Similarly $\frac{\partial K}{\partial x} = -K'$ and

$$\mathbf{B} = -\partial_x A_z \hat{\mathbf{y}} = -\frac{\mu_0 \hat{\mathbf{y}}}{2}\int_0^\infty \frac{\partial}{\partial x} K(t - x - u)\mathrm{d}u \tag{79}$$

$$= \frac{\mu_0 \hat{\mathbf{y}}}{2}\int_0^\infty K'(t_0 - u)\mathrm{d}u \tag{80}$$

$$= \frac{\mu_0 \hat{\mathbf{y}}}{2}[-K(t_0 - u)]_0^\infty \tag{81}$$

$$= \frac{\mu_0 K(t_0)}{2}\hat{\mathbf{y}} \tag{82}$$

The Poynting vector is:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0}\left(\frac{\mu_0 K(t_0)}{2}\right)^2 \hat{\mathbf{x}} = \frac{\mu_0 [K(t_0)]^2}{4}\hat{\mathbf{x}} \tag{83}$$

The plane radiates in both directions ($\pm\hat{\mathbf{x}}$) and hence:

$$P(t) = 2 \times \frac{\mu_0 [K(t_0)]^2}{4} = \frac{\mu_0}{2}[K(t_0)]^2 \tag{84}$$

5 The Hydrogen Atom—A Fermi Problem (10 points)

In this problem, we'll work out why classical electromagnetism is at odds with the stability of the hydrogen atom, a fact that was one of the key drivers for the development of quantum mechanics.

First, we must begin with an estimate of the size of the hydrogen atom. For those who are not familiar with quantum mechanics, let me give a brief, intuitive explanation. The electron in the ground state of the hydrogen atom is bound to the proton in a Coulomb potential. It can be thought of as a cloud of charge of size r, carrying some typical momentum p. Both the position and momentum of the electron are described probabilistically, with an uncertainty Δr and Δp comparable to r and p respectively. The Heisenberg uncertainty principle states that

$$\Delta x \Delta p \gtrsim \hbar\,, \tag{85}$$

setting a lower limit on how precisely we can determine particle's position and momentum. The ground state of the hydrogen atom essentially saturates this bound, i.e. $rp \sim \hbar$.

(a) Using the uncertainty principle, estimate the size of the hydrogen atom.

The total energy of the electron is the sum of its kinetic and potential energies:

$$E = \frac{p^2}{2m} - \frac{e^2}{4\pi\epsilon_0 r} \tag{86}$$

From the uncertainty principle, we saturate the bound $p \simeq \hbar/r$. Substituting this into the energy expression:

$$E(r) \simeq \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r} \tag{87}$$

To find the stable ground state (the "size" of the atom), we minimize E with respect to r:

$$\frac{\mathrm{d}E}{\mathrm{d}r} = -\frac{\hbar^2}{mr^3} + \frac{e^2}{4\pi\epsilon_0 r^2} = 0 \implies r_B = \frac{4\pi\epsilon_0\hbar^2}{me^2} \tag{88}$$

This is the **Bohr radius**, which is approximately 0.5×10^{-10} m (or 0.5 Å). This represents the scale at which quantum effects prevent the electron from collapsing into the nucleus.

(b) Now, forget about quantum mechanics, and imagine a point-like electron at r, with some initial momentum p in a circular orbit. Using classical electromagnetism, explain why the electron will spiral inward, and estimate the time taken for this to happen. Should the hydrogen atom be stable?

Classically, an electron in a circular orbit of radius r is constantly accelerating toward the center with acceleration $a = v^2/r$. According to the Larmor formula derived previously:

$$P = \frac{\mu_0 e^2 a^2}{6\pi c} \tag{89}$$

Because the electron is constantly radiating energy, its total mechanical energy $E = -e^2/(8\pi\epsilon_0 r)$ must decrease (become more negative). This implies r must decrease, causing the electron to spiral into the proton.
To estimate the lifetime τ, we note $P = -\mathrm{d}E/\mathrm{d}t$. Using $a = F/m = e^2/(4\pi\epsilon_0 mr^2)$:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} \implies \left(\frac{e^2}{8\pi\epsilon_0 r^2}\right)\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{\mu_0 e^2}{6\pi c}\left(\frac{e^2}{4\pi\epsilon_0 mr^2}\right)^2 \tag{90}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{\mu_0 e^4}{12\pi^2 c\epsilon_0 m^2 r^2} \tag{91}$$

Rearranging and integrating from $r = r_B$ to $r = 0$:

$$\tau = \frac{12\pi^2 c\epsilon_0 m^2}{\mu_0 e^4}\int_{r_B}^{0} r^2 \mathrm{d}r \tag{92}$$

$$\tau = \frac{4\pi^2\epsilon_0 c}{\mu_0}\frac{m^2 r_B^3}{e^4} \tag{93}$$

Plugging in values ($m \sim 10^{-30}$ kg, $r_B \sim 10^{-10}$ m, $e \sim 10^{-19}$ C, $\epsilon_0 \sim \times 10^{-11}$ F/m, $\mu_0 \sim 10^{-6}$ T $\cdot$ m/A, $c \sim 10^8$ m/s):

$$\tau \sim \frac{10 \cdot 10^{-11} \cdot 10^8}{10^{-6}}\frac{10^{-60} \cdot 10^{-30}}{10^{-76}} \sim 10^{-10}\mathrm{s} \tag{94}$$

Classically, the hydrogen atom should collapse in about a ten-billionth of a second. Since matter is clearly stable, classical electromagnetism fails to describe the atom, necessitating quantum mechanics.