Problem Set 8: Potentials II

1 More Infinite Wire Fields (20 points)

Consider the infinite wire as we did in class, aligned along the z-axis.

(a) Suppose the wire carries a linearly increasing current after $t = 0$, i.e.

$$I(t) = \begin{cases} 0\,, & t < 0\,, \\ kt\,, & t \geq 0\,, \end{cases} \tag{1}$$

for some constant k, everywhere along the wire. Find the electric and magnetic fields generated.

We can go about this problem similar to the example in the notes. Let's use $c = 1$, so that we don't have to carry the factor around. Using the Lorenz gauge, and noting that there is no net charge anywhere, we have $V = 0$. The current density corresponding to the current is:

$$\mathbf{J}(\mathbf{r}, t) = \frac{\delta(s)}{2\pi s} I(t)\,\hat{\mathbf{z}}\,. \tag{2}$$

Again, we can use cylindrical symmetry to assert that the vector potential is the same at all z and evaluate at $z = 0$:

$$\mathbf{A}(s, t) = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \mathrm{d}z' \int_0^{\infty} \mathrm{d}s' \frac{2\pi s'}{\sqrt{s^2 + z'^2}} \frac{\delta(s')}{2\pi s'} I(t_r)\,\hat{\mathbf{z}} \tag{3}$$

$$= \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}} \int_{-\infty}^{\infty} \mathrm{d}z' \,\frac{I(t_r)}{\sqrt{s^2 + z'^2}} \tag{4}$$

Similar to the problem in the notes, $I(t_r < 0) = 0$ and hence, the limits of the above integral should be:

$$t - \sqrt{s^2 + z'^2} > 0 \implies z'^2 \leq t^2 - s^2 \tag{5}$$

and for $s > t$,

$$\mathbf{A}(s, t) = \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}} \int_{-\sqrt{t^2 - s^2}}^{\sqrt{t^2 - s^2}} \mathrm{d}z' \,\frac{kt_r}{\sqrt{s^2 + z'^2}} \tag{6}$$

$$= \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}} \int_{-\sqrt{t^2 - s^2}}^{\sqrt{t^2 - s^2}} \mathrm{d}z' \,\frac{k(t - \sqrt{s^2 + z'^2})}{\sqrt{s^2 + z'^2}} \tag{7}$$

$$= \frac{\mu_0 k}{2\pi} \left[t \ln \left(\frac{t + \sqrt{t^2 - s^2}}{s} \right) - \sqrt{t^2 - s^2} \right] \hat{\mathbf{z}}\,. \tag{8}$$

The fields are given by:

$$\mathbf{E} = -\partial_t \mathbf{A} = \frac{\mu_0 k}{2\pi} \ln \left(\frac{t + \sqrt{t^2 - s^2}}{s} \right) \tag{9}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = -\partial_s A_z\,\hat{\phi} = \frac{\mu_0 k}{2\pi} \frac{\sqrt{t^2 - s^2}}{s}\,\hat{\phi}\,. \tag{10}$$

(b) Do the same for the case of a sudden burst of current,

$$I(t) = q_0 \delta(t)\,, \tag{11}$$

where δ is the Dirac delta function.

Here again, for $s > t$, we can use the expression:

$$\mathbf{A}(s,t) = \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}}\int_{-\infty}^{\infty} \mathrm{d}z'\,\frac{I(t_r)}{\sqrt{s^2+z'^2}} \tag{12}$$
$$= \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}}\int_{-\infty}^{\infty} \mathrm{d}z'\,\frac{q_0\delta(t-\sqrt{s^2+z'^2})}{\sqrt{s^2+z'^2}} \tag{13}$$

In order to proceed, we need to use the an identity to simplify the delta function, which for a general function $f(x)$ is:

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|} \tag{14}$$

where f' is the derivative of f and x_i are the solutions of $f(x) = 0$. For the delta function we have, $f'(z') = \frac{|z'|}{\sqrt{s^2+z'^2}}$ and $z'_i = \pm\sqrt{t^2-s^2}$. Using the identity above in the integral, we get:

$$\mathbf{A}(s,t) = \frac{\mu_0 q_0}{2\pi}\,\hat{\mathbf{z}}\,\frac{1}{\sqrt{t^2-s^2}} \tag{15}$$

The fields are given by:

$$\mathbf{E} = -\partial_t\mathbf{A} = \frac{\mu_0 q_0}{2\pi}\frac{t}{(t^2-s^2)^{3/2}}) \tag{16}$$
$$\mathbf{B} = -\partial_s A_z\,\hat{\phi} = \frac{\mu_0 q_0}{2\pi}\frac{s}{(t^2-s^2)^{3/2}}\,\hat{\phi}\,. \tag{17}$$

2 Electrostatic and Magnetostatic Approximations (15 points)

In class, I throw around the words "electrostatic" and "magnetostatic" quite freely. In this question, we'll consider what these limits actually mean by showing how the Jefimenko equations reduce to Coulomb's law and the Biot-Savart law.

(a) By the "electrostatic regime", we really mean $\dot{\mathbf{J}} = 0$ everywhere, where the dot represents a time derivative.

i) Show that if $\dot{\mathbf{J}} = 0$, $\rho(\boldsymbol{r},t) = \rho(\boldsymbol{r},0) + \dot{\rho}(\boldsymbol{r},0)t$.

ii) Show that

$$\mathbf{E}(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3\boldsymbol{r}'\rho(\boldsymbol{r}',t)\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,, \tag{18}$$

which is simply Coulomb's law. Note that the charge density inside the integral is evaluated at t, and *not* the retarded time.

Let's start by taking the time derivative of the continuity equation:

$$\ddot{\rho} = \partial_t(\nabla\cdot\mathbf{J}) = \nabla.\dot{\mathbf{J}} = 0 \tag{19}$$

Integrating the above equation twice, we the first expression:

$$\dot{\rho}(\mathbf{r},t) = \rho(\mathbf{r},0) \tag{20}$$
$$\rho(\mathbf{r},t) = \rho(\mathbf{r},0) + t\rho(\mathbf{r},0) \tag{21}$$

Now consider the Jefimenko equation. The last term drops out, since $\dot{\mathbf{J}} = 0$. We can then use the two equations above to get:

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3\mathbf{r}' \left(\rho(\mathbf{r}, t_r) \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} + \dot{\rho}(\mathbf{r}, t_r) \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2} \right) \tag{22}$$

$$= \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3\mathbf{r}' \left((\rho(\mathbf{r}, t) - |\mathbf{r} - \mathbf{r}'|\dot{\rho}(\mathbf{r}, t)) \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} + \dot{\rho}(\mathbf{r}, t) \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2} \right) \tag{23}$$

$$= \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3\mathbf{r}' \, \rho(\mathbf{r}, t) \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \tag{24}$$

(b) By the "magnetostatic regime", we mean that the current density changes slowly enough such that we can expand the current density evaluated at the retarded time as

$$\mathbf{J}(\boldsymbol{r}, t_r) \simeq \mathbf{J}(\boldsymbol{r}, t) + (t_r - t)\dot{\mathbf{J}}(\boldsymbol{r}, t) \,, \tag{25}$$

and we can neglect all higher derivatives in this expansion. Show that in this limit,

$$\mathbf{B}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3\boldsymbol{r}' \, \frac{\mathbf{J}(\boldsymbol{r}', t)}{|\boldsymbol{r} - \boldsymbol{r}'|^2} \times \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|} \,, \tag{26}$$

which is the Biot-Savart law. Note once again that we evaluate the currents at time t inside the integral. Notice that the magnetostatic regime has a much weaker requirement than the electrostatic regime! We can still use the Biot-Savart law even with a nonzero $\dot{\mathbf{J}}$.

In the limit

$$\mathbf{J}(\boldsymbol{r}, t_r) \simeq \mathbf{J}(\boldsymbol{r}, t) + (t_r - t)\dot{\mathbf{J}}(\boldsymbol{r}, t) \tag{27}$$

$$= \mathbf{J}(\boldsymbol{r}, t) - |\mathbf{r} - \mathbf{r}'| \, \dot{\mathbf{J}}(\boldsymbol{r}, t) \,, \tag{28}$$

we have

$$\dot{\mathbf{J}}(\boldsymbol{r}, t_r) \simeq \dot{\mathbf{J}}(\boldsymbol{r}, t) + (t_r - t)\ddot{\mathbf{J}}(\boldsymbol{r}, t) \simeq \dot{\mathbf{J}}(\boldsymbol{r}, t) \,. \tag{29}$$

Substituting these in the Jefimenko equation, we get:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \mathrm{d}^3\mathbf{r}' \left(\frac{\mathbf{J}(\mathbf{r}, t_r)}{|\mathbf{r} - \mathbf{r}'|^2} + \frac{\dot{\mathbf{J}}(\mathbf{r}, t_r)}{|\mathbf{r} - \mathbf{r}'|} \right) \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \tag{30}$$

$$= \frac{\mu_0}{4\pi} \int \mathrm{d}^3\mathbf{r}' \left(\frac{\mathbf{J}(\boldsymbol{r}, t) - |\mathbf{r} - \mathbf{r}'| \, \dot{\mathbf{J}}(\boldsymbol{r}, t)}{|\mathbf{r} - \mathbf{r}'|^2} + \frac{\dot{\mathbf{J}}(\mathbf{r}, t)}{|\mathbf{r} - \mathbf{r}'|} \right) \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \tag{31}$$

$$= \frac{\mu_0}{4\pi} \int \mathrm{d}^3\mathbf{r}' \, \frac{\mathbf{J}(\boldsymbol{r}, t)}{|\mathbf{r} - \mathbf{r}'|^2} \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \tag{32}$$

3 Another Derivation of the Jefimenko Equations (15 points)

(a) Starting from Maxwell's equations, show that

$$\begin{aligned} \Box\mathbf{E} &= \frac{1}{\epsilon_0}\nabla\rho + \mu_0 \frac{\partial \mathbf{J}}{\partial t} \,, \\ \Box\mathbf{B} &= -\mu_0(\nabla \times \mathbf{J}) \,. \end{aligned} \tag{33}$$

Taking the curl of Faraday's law and using the Ampere-Maxwell equation, we get:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \tag{34}$$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}\left(\mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \tag{35}$$

Now, we can use Gauss law and rearrange to arrive at:

$$\nabla^2 \mathbf{E} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\epsilon_0}\nabla\rho + \mu_0 \frac{\partial \mathbf{J}}{\partial t} \tag{36}$$

If we instead start with the curl of the Ampere-Maxwell equation and use Faraday's law:

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0(\nabla \times \mathbf{J}) + \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) \tag{37}$$

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \mu_0(\nabla \times \mathbf{J}) - \mu_0 \epsilon_0 \frac{\partial}{\partial t}\frac{\partial \mathbf{B}}{\partial t} \tag{38}$$

(b) What are the solutions to the inhomogeneous wave equations shown above?

In general, the solution to the inhomogenous equation $\Box \mathbf{F} = \mathbf{f}(\mathbf{r}, t)$ is the expression:

$$\mathbf{F}(\mathbf{r}, t) = -\frac{1}{4\pi}\int \frac{\mathbf{f}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \tag{39}$$

Using this for the expression from part(a), we get:

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi}\int d^3\mathbf{r}' \left(\frac{-1}{\epsilon_0}\frac{(\nabla\rho)(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} - \mu_0 \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|}\right) \tag{40}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int d^3\mathbf{r}' \frac{(\nabla \times \mathbf{J})(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \tag{41}$$

(c) Prove the following useful identities:

$$\begin{aligned}
\nabla'[\rho(\boldsymbol{r}', t_r)] &= (\nabla\rho)(\boldsymbol{r}', t_r) + \frac{\partial \rho(\boldsymbol{r}', t_r)}{\partial t_r}\nabla' t_r\,, \\
\nabla' t_r &= \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|}\,, \\
(\nabla \times \mathbf{J})(\boldsymbol{r}', t_r) &= \nabla' \times \mathbf{J}(\boldsymbol{r}', t_r) + \dot{\mathbf{J}}(\boldsymbol{r}', t_r) \times \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|}\,.
\end{aligned} \tag{42}$$

The first expression can be obtained by using the multivariable chain rule:

$$\nabla' \rho(\mathbf{r}', t_r) = (\nabla' \rho(\mathbf{r}', t))|_{t = t_r} + \frac{\partial \rho(\mathbf{r}', t_r)}{\partial t_r}\nabla' t_r \tag{43}$$

$$= (\nabla\rho)(\mathbf{r}', t_r) + \frac{\partial \rho(\mathbf{r}', t_r)}{\partial t_r}\nabla' t_r \tag{44}$$

For the second expression:

$$\nabla' t_r = \nabla'(t - |\mathbf{r} - \mathbf{r}'|) = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \tag{45}$$

For the third expression, we again need to use the chain rule:

$$\nabla' \times \mathbf{J}(\boldsymbol{r}', t_r) = (\nabla \times \mathbf{J})(\boldsymbol{r}', t)|_{t=t_r} + \nabla' t_r \times \partial_{t_r} \mathbf{J}(\boldsymbol{r}', t_r) \tag{46}$$

$$= (\nabla \times \mathbf{J})(\boldsymbol{r}', t_r) - \dot{\mathbf{J}}(\boldsymbol{r}', t_r) \times \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|}\,. \tag{47}$$

(d) Perform integration by parts of the solutions in part (b) to recover Jefimenko's equations for **E** and **B**.

Using the identities from part(c) in the expression for **E** in part (b), we get:

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi} \int d^3\mathbf{r}' \left(\frac{-1}{\epsilon_0} \frac{(\nabla\rho)(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} - \mu_0 \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \right) \tag{48}$$

$$= \frac{1}{4\pi\epsilon_0} \int d^3\mathbf{r}' \left(-\frac{\left(\nabla'\rho(\mathbf{r}', t_r) - \frac{\partial\rho(\mathbf{r}', t_r)}{\partial t_r} \nabla' t_r\right)}{|\mathbf{r} - \mathbf{r}'|} - \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \right) \tag{49}$$

$$= \frac{1}{4\pi\epsilon_0} \int d^3\mathbf{r}' \left(-\frac{\nabla'\rho(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} + \dot{\rho}(\mathbf{r}, t) \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2} - \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \right) \tag{50}$$

Now, we use integration by part for the first term:

$$\int d^3\mathbf{r}' \frac{\nabla'\rho(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} = \int d^3\mathbf{r}'\, \nabla' \frac{\rho(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} - \int d^3\mathbf{r}'\, \rho(\mathbf{r}', t_r) \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \tag{51}$$

$$= 0 + \int d^3\mathbf{r}'\, \rho(\mathbf{r}', t_r) \frac{1}{|\mathbf{r} - \mathbf{r}'|^2} \nabla' |\mathbf{r} - \mathbf{r}'| \tag{52}$$

$$= -\int d^3\mathbf{r}'\, \rho(\mathbf{r}', t_r) \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \tag{53}$$

Substituting this back into the above expression we get the Jefimenko equation for **E**.
We use the same procedure for the expression for **B** in part(b):

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int d^3\mathbf{r}' \frac{(\nabla \times \mathbf{J})(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \tag{54}$$

$$= \frac{\mu_0}{4\pi} \int d^3\mathbf{r}' \frac{\left(\nabla' \times \mathbf{J}(\boldsymbol{r}', t_r) + \dot{\mathbf{J}}(\boldsymbol{r}', t_r) \times \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|}\right)}{|\mathbf{r} - \mathbf{r}'|} \tag{55}$$

$$= \frac{\mu_0}{4\pi} \int d^3\mathbf{r}' \left(\frac{\nabla' \times \mathbf{J}(\boldsymbol{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} + \frac{\dot{\mathbf{J}}(\boldsymbol{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \times \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) \tag{56}$$

Using integration by parts in the first term here:

$$\int d^3\mathbf{r}' \frac{\nabla' \times \mathbf{J}(\boldsymbol{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} = \int d^3\mathbf{r}'\ \nabla' \times \frac{\mathbf{J}(\boldsymbol{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} - \int d^3\mathbf{r}' \left(\nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) \times \mathbf{J}(\boldsymbol{r}', t_r) \tag{57}$$

$$= 0 - \int d^3\mathbf{r}' \left(\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) \times \mathbf{J}(\boldsymbol{r}', t_r) \tag{58}$$

$$= \int d^3\mathbf{r}'\, \mathbf{J}(\boldsymbol{r}', t_r) \times \left(\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) \tag{59}$$

Substituting this back into the above expression we get the Jefimenko equation for **B**.

4 Point Charge Field (10 points)

I know I said we wouldn't do point charges, but there's nothing stopping us from looking at them in your problem set! The solution can basically be found in the discussion in Griffiths 10.3, but the goal here is to process it on your own.

Let me just give you the point charge fields. Consider a charge located at position $\boldsymbol{r}'(t)$ at some time t. Let's also define its velocity $\mathbf{v}(t) \equiv \mathrm{d}\boldsymbol{r}'/\mathrm{d}t$ and acceleration $\mathbf{a}(t) \equiv \mathrm{d}\boldsymbol{v}/\mathrm{d}t$. The electric field is given by

$$\mathbf{E}(\boldsymbol{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{[(\boldsymbol{r} - \boldsymbol{r}') \cdot \mathbf{u}]^3} \left[(1 - v^2)\mathbf{u} + (\boldsymbol{r} - \boldsymbol{r}') \times (\mathbf{u} \times \mathbf{a})\right] , \tag{60}$$

with

$$\mathbf{u} = \left[\frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|} - \mathbf{v}\right] \tag{61}$$

where I should remind you that everything in square brackets should be evaluated at the retarded time, defined as

$$t_r = t - |\boldsymbol{r} - \boldsymbol{r}'(t_r)| \, . \tag{62}$$

This looks very confusing, but here's how to understand it: as the charge moves around on its trajectory, if you want to find the field $\mathbf{E}(\boldsymbol{r}, t)$, that field is determined by the charge position, velocity and acceleration at a single point on the trajectory $\boldsymbol{r}'(t_r)$, whose separation from $\boldsymbol{r}$ is exactly the speed of light times the time interval $t - t_r$ (you can show that there is only one such point, because the charge has to travel at less than the speed of light).

The magnetic field is given by

$$\mathbf{B}(\boldsymbol{r}, t) = \left[\frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|}\right] \times \mathbf{E}(\boldsymbol{r}, t) \, , \tag{63}$$

where again we evaluate the unit vector at the retarded time (it points to where the charge was at t_r and *not* at t).

(a) Determine the $\mathbf{E}$ and $\mathbf{B}$ fields for $\mathbf{v} = 0$ and $\mathbf{a} = 0$.

> If the charge is stationary, then
>
> $$\mathbf{u} = \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{64}$$
>
> and
>
> $$\mathbf{E}(\boldsymbol{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{[|\boldsymbol{r} - \boldsymbol{r}'|]}{\left[(\boldsymbol{r} - \boldsymbol{r}') \cdot \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|}\right]^3} \left[\frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|} + 0\right] \tag{65}$$
>
> $$= \frac{q}{4\pi\epsilon_0} \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \, . \tag{66}$$
>
> The magnetic field is simply $\mathbf{B} = \mathbf{v} \times \mathbf{E} = 0$. This is exactly what you'd expect for a stationary charge.

(b) For a charge moving with constant velocity, determine the $\mathbf{E}$ and $\mathbf{B}$ fields as a function of $\mathbf{R} \equiv \boldsymbol{r} - \mathbf{v}t$ (i.e. the separation $|\boldsymbol{r} - \boldsymbol{r}'|$ at time t) and θ, the angle between $\mathbf{R}$ and $\mathbf{v}$.

With $a = 0$, we get:

$$\mathbf{E}(\boldsymbol{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{[|\boldsymbol{r} - \boldsymbol{r}'|]}{[(\boldsymbol{r} - \boldsymbol{r}') \cdot \mathbf{u}]^3} (1 - v^2) [\mathbf{u}] \ , \tag{67}$$

Without loss of generality we assume that $\mathbf{r}'(t) = \mathbf{v}t$. The retarded time is now given by:

$$t_r = t - |\mathbf{r} - \mathbf{v}t_r| = t - |\mathbf{R}_r| \tag{68}$$

where we define $\mathbf{R}_r \equiv \mathbf{v}t_r = [\mathbf{r} - \mathbf{r}']$. From the above expression, we get the relation:

$$\begin{aligned} \mathbf{R}_r &= \mathbf{r} - \mathbf{v}t + \mathbf{v}(t - t_r) && (69) \\ &= \mathbf{R} + \mathbf{v}|\mathbf{R}_r| && (70) \end{aligned}$$

We can rewrite the retarded time expression for $\mathbf{u}$ as:

$$\begin{aligned} [\mathbf{u}] &= \left[\frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|} \right] - \mathbf{v} && (71) \\ &= \frac{\mathbf{R}_r}{|\mathbf{R}_r|} - \mathbf{v} && (72) \end{aligned}$$

Substituting this back into our expression for $\mathbf{E}$, we get:

$$\begin{aligned} \mathbf{E}(\boldsymbol{r}, t) &= \frac{q}{4\pi\epsilon_0} \frac{(1 - v^2)}{(|\mathbf{R}_r| - \mathbf{R}_r \cdot \mathbf{v})^3} (\mathbf{R}_r - \mathbf{v}|\mathbf{R}_r|) && (73) \\ &= \frac{q}{4\pi\epsilon_0} \frac{(1 - v^2)}{(|\mathbf{R}_r| - \mathbf{R}_r \cdot \mathbf{v})^3} \mathbf{R} && (74) \end{aligned}$$

In order to evaluate the denominator $\mathbf{R}_r \cdot \mathbf{u} = |\mathbf{R}_r| - \mathbf{R}_r \cdot \mathbf{v}$, we need to do some non-intuitive algebraic manipulations which can replace $\mathbf{R}_r$ with $\mathbf{R}$. We begin by taking the norm of the definition that relates them:

$$\begin{aligned} \mathbf{R}^2 &= \mathbf{R}_r^2 + v^2 \mathbf{R}_r^2 - 2\, \mathbf{v} \cdot \mathbf{R} && (75) \\ &= (|\mathbf{R}_r| - \mathbf{R}_r \cdot \mathbf{v})^2 + v^2 \mathbf{R}_r^2 - (\mathbf{v} \cdot \mathbf{R}_r)^2 && (76) \\ &= (|\mathbf{R}_r| - \mathbf{R}_r \cdot \mathbf{v})^2 + (\mathbf{v} \times \mathbf{R_r})^2 && (77) \end{aligned}$$

where in the last step, we used the fact that for any angle θ_r between $\mathbf{v}$ and $\mathbf{R}_r$, $1 - \cos^2 \theta_r = \sin^2 \theta_r$. Starting from the relation between $\mathbf{R}_r$ with $\mathbf{R}$, and taking a cross product with $\mathbf{v}$, we see:

$$\mathbf{v} \times \mathbf{R}_r = \mathbf{v} \times \mathbf{R} \tag{78}$$

Substituting this back into the expression above, we get:

$$\begin{aligned} (|\mathbf{R}_r| - \mathbf{R}_r \cdot \mathbf{v})^2 &= \mathbf{R}^2 - (\mathbf{v} \times \mathbf{R})^2 && (79) \\ &= \mathbf{R}^2 (1 - v^2 \sin^2 \theta) && (80) \end{aligned}$$

Plugging this into our expression for $\mathbf{E}$, we arrive at:

$$\begin{aligned} \mathbf{E}(\boldsymbol{r}, t) &= \frac{q}{4\pi\epsilon_0} \frac{(1 - v^2)}{(|\mathbf{R}_r| - \mathbf{R}_r \cdot \mathbf{v})^3} (\mathbf{R}_r - \mathbf{v}|\mathbf{R}_r|) && (81) \\ &= \frac{q}{4\pi\epsilon_0} \frac{(1 - v^2)}{(1 - v^2 \sin^2 \theta)^{3/2}} \frac{\mathbf{R}}{\mathbf{R}^3} && (82) \end{aligned}$$

The magnetic field is:

$$\mathbf{B}(\boldsymbol{r},t) = \left[\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\right] \times \mathbf{E}(\boldsymbol{r},t) \tag{83}$$
$$= \frac{\mathbf{R}_r}{|\mathbf{R}_r|} \times \mathbf{E}(\boldsymbol{r},t) \tag{84}$$
$$= \left(\frac{\mathbf{R}}{|\mathbf{R}_r|} + \mathbf{v}\right) \times \mathbf{E}(\boldsymbol{r},t) \tag{85}$$
$$= \mathbf{v} \times \mathbf{E}(\boldsymbol{r},t) \tag{86}$$

where we used the fact that $\mathbf{E}$ points along $\mathbf{R}$ for the last simplification.

(c) For the fields in the previous part, what is the result in the limit where $v \ll 1$?

When $v \ll 1$, we can drop the factors of v^2 and get:

$$\mathbf{E}(\boldsymbol{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{R}}{\mathbf{R}^3} \tag{87}$$
$$\mathbf{B}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{v}\times\mathbf{R}}{\mathbf{R}^3} \tag{88}$$

which are expressions we've seen before for Coulombs law and Biot-Savart law.

(d) Consider a particle of charge q traveling at constant speed v along the x-axis. Calculate the total power passing through the plane $x = a$, at the moment the particle itself is at the origin.

In order to calculate the power though the plane $x = a$, we need to compute the integral of $\mathbf{S}\cdot\hat{\mathbf{x}}$ on the plane. The Poynting vector can be computed as:

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{89}$$
$$= \frac{1}{\mu_0}\mathbf{E}\times(\mathbf{v}\times\mathbf{E}) \tag{90}$$
$$= \frac{1}{\mu_0}(E^2\mathbf{v} - (\mathbf{E}.\mathbf{v})\mathbf{E}) \tag{91}$$

Note that $\mathbf{v} = v\,\hat{\mathbf{x}}$ and hence

$$\mathbf{S}\cdot\hat{\mathbf{x}} = \frac{1}{\mu_0}(E^2 v - vE_x^2) \tag{92}$$
$$= \epsilon_0 v(E_y^2 + E_z^2) \tag{93}$$

For a point on the plane on $x = a$, $\mathbf{r} = (a, y, z)$. When the charge is at the origin, $t = 0$ and $\mathbf{R} = (a, y, z)$. Notice that on the 2D plane, we can use cylindrical symmetry to write out

$$\mathbf{R}^2 = a^2 + (y^2 + z^2) = a^2 + \rho^2\,. \tag{94}$$

Also,

$$\sin\theta = \frac{\rho}{|\mathbf{R}|} = \frac{\rho}{\sqrt{a^2+\rho^2}} \tag{95}$$

With this, the electric field on the plane is given by:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{1-v^2}{(1-v^2\sin^2\theta)^{3/2}} \frac{(a,y,z)}{(a^2+\rho^2)^{3/2}} \tag{96}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1-v^2}{(a^2+\rho^2(1-v^2))^{3/2}} (a,y,z) \tag{97}$$

The power is now given by :

$$P = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \mathbf{S} \cdot \hat{\mathbf{x}} \tag{98}$$

$$= \int_0^{\infty} 2\pi\rho\, \mathrm{d}\rho\, \epsilon_0 v \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{(1-v^2)^2}{(a^2+\rho^2(1-v^2))^3} \rho^2 \tag{99}$$

Using $u = \rho^2$,

$$P = \frac{vq^2(1-v^2)^2}{16\pi\epsilon_0} \int_0^{\infty} \mathrm{d}u \frac{u}{(a^2+u(1-v^2))^3} \tag{100}$$

$$= \frac{vq^2(1-v^2)^2}{16\pi\epsilon_0} \frac{1}{2(1-v^2)^2 a^2} \tag{101}$$

$$= \frac{q^2 v}{32\pi\epsilon_0 a^2} \tag{102}$$