Problem Set 7: Potentials I

1 Baby Natural Units Practice (5 points)

(a) Consider a plane EM wave in vacuum. Using natural units, show that the energy density associated with the electric field is equal to the energy density associated with the magnetic field at every point in space and time.

For a plane EM wave,

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}.\mathbf{r}-\omega t)} \,, \tag{1}$$
$$\mathbf{B} = \mathbf{E}_0 e^{i(\mathbf{k}.\mathbf{r}-\omega t)} \,, \tag{2}$$

in natural units. The energy density is

$$u = \frac{1}{2}\left(\epsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2\right) \,, \tag{3}$$
$$= \frac{\epsilon_0}{2}\left(\mathbf{E}_0^2 + \mathbf{E}_0^2\right) \,. \tag{4}$$

The energy density associated with the electric and the magnetic field are equal to $\frac{\epsilon_0}{2}\mathbf{E}_0^2$ in the natural units.

(b) In natural units, mass, energy and momentum all have the same units. Evaluate the electron mass in SI units (i.e. kilograms), given that its mass in natural units is 511 keV.

Electron mass is in the baby units is

$$m = 511\,\mathrm{keV} = 511 \times 10^3 \times 1.6 \times 10^{-19}\,\mathrm{J} \tag{5}$$
$$= 8.176 \times 10^{-14}\,\mathrm{J} = 8.176 \times 10^{-14}\,\mathrm{kgm^2s^{-2}} \tag{6}$$

To convert this to SI units, we need to divide by c^2:

$$m = 9.1 \times 10^{-31}\,\mathrm{kg} \,. \tag{7}$$

(c) Evaluate the classical electron radius r_e, defined in natural units as

$$r_e = \frac{e^2}{4\pi\epsilon_0 m_e} \,. \tag{8}$$

Plugging in numbers:

$$r_e = \frac{e^2}{4\pi\epsilon_0 m_e} = \frac{(1.6 \times 10^{-19}\mathrm{C})^2}{4 \times 3.14 \times (8.85 \times 10^{-12}\,\mathrm{m^{-3}kg^{-1}s^2C^2})(9.1 \times 10^{-31}\mathrm{kg})} \tag{9}$$
$$= 2.5 \times 10^2\,\mathrm{m^2s^{-2}} \tag{10}$$

Again, we need to divide by c^2 to get the right units for the radius:

$$r_e = 2.8 \times 10^{-15}\,\mathrm{m} \tag{11}$$

(d) The total power radiated by a nonrelativistic point charge with charge q moving with acceleration a is given by the Larmor formula, which in $c = 1$ units can be written as

$$P = \frac{q^2 a^2}{6\pi\epsilon_0}\,. \tag{12}$$

Determine the power radiated in watts by an electron accelerating at $1\,\mathrm{m\,s^{-2}}$.

Here,

$$P = \frac{q^2 a^2}{6\pi\epsilon_0} = \frac{(1.6\times 10^{-19}\mathrm{C})^2 (1\,\mathrm{ms^{-1}})^2}{6\times 3.14\times (8.85\times 10^{-12}\,\mathrm{m^{-3}kg^{-1}s^2C^2})} \tag{13}$$

$$= 1.5\times 10^{-28}\mathrm{kgm^5s^{-4}} \tag{14}$$

In this case, we divide by c^3 to get the right units for power to obtain

$$P = 5.7\times 10^{-54}\,\mathrm{W}\,. \tag{15}$$

Note that the remainder of this problem set will be in natural units.

2 Plastic Ring of Charge (15 points)

Suppose you take a plastic ring of radius a and glue charge on it, so that the line charge density is given by $\lambda_0|\sin(\theta/2)|$ (note the absolute value). Then, you spin the loop about its axis at an angular velocity ω. Find the exact scalar and vector potentials and the center of the ring. You should find

$$\mathbf{A} = \frac{\mu_0\lambda_0\omega a}{3\pi}\left[\sin(\omega(t-a))\hat{\mathbf{x}} - \cos(\omega(t-a))\hat{\mathbf{y}}\right]\,. \tag{16}$$

For this problem, we shall start by working in cylindrical coordinates. Due to the rotation, the linear charge density at a later time is given by

$$\lambda(\mathbf{r}, t) = \lambda_0\left|\sin\left(\frac{\theta-\omega t}{2}\right)\right|\,. \tag{17}$$

This corresponds to a charge density

$$\rho(\mathbf{r}, t) = \lambda(\mathbf{r}, t)\ \delta(z)\delta(r-a)\,. \tag{18}$$

Since the charge element $dq = \lambda dl$ on the loop is moving at a velocity $v = a\omega$, the current density is given by:

$$\mathbf{J}(\mathbf{r}, t) = \lambda(\mathbf{r}, t)a\omega\,\delta(z)\delta(r-a)\,\hat{\theta} \tag{19}$$

The generic formula for the scalar and vector potential (in natural units) is:

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3r'\,\frac{\rho(\mathbf{r}, t-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} \tag{20}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3r'\,\frac{\mathbf{J}(\mathbf{r}, t-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} \tag{21}$$

At the center of the loop, the saclar potential is:

$$V(0,t) = \frac{1}{4\pi\epsilon_0}\int_0^\infty \mathrm{d}r \int_{-\infty}^{\infty}\mathrm{d}z \int_0^{2\pi} a\mathrm{d}\theta\, \frac{\lambda_0 \left|\sin\left(\frac{\theta-\omega(t-a)}{2}\right)\right| \delta(z)\delta(r-a)}{a} \tag{22}$$

$$= \frac{\mu_0\lambda_0}{4\pi}\int_0^{2\pi}\mathrm{d}\theta \left|\sin\left(\frac{\theta}{2}\right)\right| \tag{23}$$

$$= \frac{\mu_0\lambda_0}{\pi} \tag{24}$$

The vector potential at the center is:

$$\mathbf{A}(0,t) = \frac{\mu_0}{4\pi}\int_0^\infty \mathrm{d}r \int_{-\infty}^{\infty}\mathrm{d}z \int_0^{2\pi} a\mathrm{d}\theta\, \frac{\lambda_0 \left|\sin\left(\frac{\theta-\omega(t-a)}{2}\right)\right| a\omega\delta(z)\delta(r-a)\,\hat{\theta}}{a} \tag{25}$$

$$= \frac{\mu_0\lambda_0\omega a}{4\pi}\int_0^{2\pi}\mathrm{d}\theta \left|\sin\left(\frac{\theta-\omega(t-a)}{2}\right)\right| (-\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}) \tag{26}$$

$$= \frac{\mu_0\lambda_0\omega a}{4\pi}\left[\frac{4}{3}\sin(\omega(t-a))\hat{\mathbf{x}} - \frac{4}{3}\cos(\omega(t-a))\hat{\mathbf{y}}\right] \tag{27}$$

$$= \frac{\mu_0\lambda_0\omega a}{3\pi}\left[\sin(\omega(t-a))\hat{\mathbf{x}} - \cos(\omega(t-a))\hat{\mathbf{y}}\right] \tag{28}$$

3 Transforming into Lorenz Gauge (5 points)

Starting from some initial V and $\mathbf{A}$, show that it is always possible to perform a gauge transformation to V' and $\mathbf{A}'$ such that

$$\nabla\cdot\mathbf{A} = -\frac{\partial V}{\partial t}\,, \tag{29}$$

i.e. it is always possible to fix the gauge to be Lorenz gauge. (*Hint:* At some point, you should find that you can always go into Lorenz gauge, if you know how to solve the inhomogeneous wave equation. But you do know how to solve the inhomogeneous wave equation! We did that in class.)

If we start with V and $\mathbf{A}$, and perform gauge transformation

$$V' = V - \partial_t\lambda \quad \text{and} \quad \mathbf{A}' = \mathbf{A} + \nabla\lambda\,, \tag{30}$$

so that we satisfy,

$$\nabla\cdot\mathbf{A}' = -\partial_t V'\,, \tag{31}$$

we need to choose λ so that it satisfies the condition

$$\nabla\cdot\mathbf{A} + \nabla^2\lambda = -\partial V + \partial_t^2\lambda\,. \tag{32}$$

This condition can be rewritten as:

$$\nabla^2\lambda - \partial_t^2\lambda = -\partial V - \nabla\cdot\mathbf{A} \tag{33}$$

This is a inhomogeneous wave equation for λ, that can be solved for any given V and $\mathbf{A}$. Hence, it is always possible to perform a gauge transform into Lorenz gauge.

4 Retarded Potentials Satisfy Lorenz Gauge (15 points)

In lecture, I simply wrote down the expressions for the retarded potential, but did not actually show that they satisfy the Lorenz gauge condition, i.e. $\nabla \cdot \mathbf{A} = -\partial V/\partial t$. Show that this is indeed true. You might first want to show that for $\mathbf{J}(\boldsymbol{r}', t_r)$ where $t_r = t - |\boldsymbol{r} - \boldsymbol{r}'|$,

$$\nabla \cdot \frac{\mathbf{J}}{|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{\nabla \cdot \mathbf{J}}{|\boldsymbol{r} - \boldsymbol{r}'|} + \frac{\nabla' \cdot \mathbf{J}}{|\boldsymbol{r} - \boldsymbol{r}'|} - \nabla' \cdot \frac{\mathbf{J}}{|\boldsymbol{r} - \boldsymbol{r}'|} , \tag{34}$$

where ∇ denotes derivatives with respect to $\boldsymbol{r}$, and ∇' denotes derivatives with respect to $\boldsymbol{r}'$. Don't forget that $\mathbf{J}(\boldsymbol{r}', t_r)$ depends explicitly on $\boldsymbol{r}'$, but also implicitly on $\boldsymbol{r}'$ and $\boldsymbol{r}$ through t_r. The continuity equation should also be useful here.

Let's first prove the identity. Using the product rule, we get

$$\nabla \cdot \frac{\mathbf{J}}{|\mathbf{r} - \mathbf{r}'|} = \frac{\nabla \cdot \mathbf{J}}{|\mathbf{r} - \mathbf{r}'|} + \mathbf{J} \cdot \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} . \tag{35}$$

We can also write out

$$\nabla' \cdot \frac{\mathbf{J}}{|\mathbf{r} - \mathbf{r}'|} = \frac{\nabla' \cdot \mathbf{J}}{|\mathbf{r} - \mathbf{r}'|} + \mathbf{J} \cdot \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \tag{36}$$

$$= \frac{\nabla' \cdot \mathbf{J}}{|\mathbf{r} - \mathbf{r}'|} \nabla' \cdot \mathbf{J} - \mathbf{J} \cdot \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \tag{37}$$

Adding the two equations and rearranging, we the the required identity

$$\nabla \cdot \frac{\mathbf{J}}{|\mathbf{r} - \mathbf{r}'|} = \frac{\nabla \cdot \mathbf{J}}{|\mathbf{r} - \mathbf{r}'|} + \frac{\nabla' \cdot \mathbf{J}}{|\mathbf{r} - \mathbf{r}'|} - \nabla' \cdot \frac{\mathbf{J}}{|\mathbf{r} - \mathbf{r}'|} . \tag{38}$$

Now consider the expressions for the retarded potentials:

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3 r' \, \frac{\rho(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \tag{39}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3 r' \, \frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \tag{40}$$

The divergence of the vector potential is:

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3 r' \, \nabla \cdot \frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \tag{41}$$

We can use the identity above and note that $\int \mathrm{d}^3 r' \, \nabla' \cdot \frac{\mathbf{J}}{|\mathbf{r} - \mathbf{r}'|} = 0$ to arrive at

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3 r' \, \frac{\nabla \cdot \mathbf{J}(\mathbf{r}', t_r) + \nabla' \cdot \mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} . \tag{42}$$

Now, notice that

$$\nabla \cdot \mathbf{J}(\mathbf{r}', t_r) = \frac{\partial}{\partial t_r} \mathbf{J}(\mathbf{r}', t_r) . \nabla t_r \tag{43}$$

$$\nabla' \cdot \mathbf{J}(\mathbf{r}', t_r) = \nabla' \cdot \mathbf{J}(\mathbf{r}', t)|_{t = t_r} + \frac{\partial}{\partial t_r} \mathbf{J}(\mathbf{r}', t_r) . \nabla' t_r \tag{44}$$

$$= \nabla' \cdot \mathbf{J}(\mathbf{r}', t)|_{t = t_r} - \frac{\partial}{\partial t_r} \mathbf{J}(\mathbf{r}', t_r) . \nabla t_r \tag{45}$$

The term $\nabla' \cdot \mathbf{J}(\mathbf{r}', t)|_{t=t_r}$ should really be thought of as the vector partial derivative of $\mathbf{J}$ with respect to $\mathbf{r}'$ while treating t_r as not a function of $\mathbf{r}'$. Using these substitutions in the expression for $\nabla \cdot \mathbf{A}$ above, we get:

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3 r' \, \frac{\nabla' \cdot \mathbf{J}(\mathbf{r}', t)|_{t=t_r}}{|\mathbf{r} - \mathbf{r}'|} \,. \tag{46}$$

The time derivative of the scalar potential is:

$$\frac{\partial}{\partial t} V(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3 r' \, \frac{\frac{\partial}{\partial t}\rho(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \tag{47}$$

where we used the relation $\frac{1}{\epsilon_0} = \mu_0$ (in natural units). Finally, we can put the two expressions together and use the continuity equation to see that

$$\begin{aligned}
\nabla \cdot \mathbf{A}(\mathbf{r}, t) + \frac{\partial}{\partial t} V(\mathbf{r}, t) &= \frac{\mu_0}{4\pi} \int \mathrm{d}^3 r' \, \frac{\nabla' \cdot \mathbf{J}(\mathbf{r}', t)|_{t=t_r} + \frac{\partial}{\partial t}\rho(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} && (48) \\
&= \frac{\mu_0}{4\pi} \int \mathrm{d}^3 r' \, \frac{\left(\nabla' \cdot \mathbf{J}(\mathbf{r}', t) + \frac{\partial}{\partial t}\rho(\mathbf{r}', t)\right)|_{t=t_r}}{|\mathbf{r} - \mathbf{r}'|} && (49) \\
&= 0 && (50)
\end{aligned}$$

5 Effective Charge of an Expanding Sphere (15 points)

An expanding sphere, radius $R(t) = vt$ with $t > 0$ and constant v, carries a charge Q, uniformly distributed over its volume. We can define the effective charge measured at the origin as the integral over the charge density, evaluated at the retarded time with respect to the origin:

$$Q_{\text{eff}}(t) = \int d^3\boldsymbol{r}' \, \rho(\boldsymbol{r}', t_r) \,. \tag{51}$$

Show that Q_{eff} is actually independent of time, and that

$$Q_{\text{eff}} = Q \left(1 - \frac{3}{4} v\right) \tag{52}$$

in natural units, if $v \ll 1$.

The charge density is given by

$$\rho(t) = \frac{Q}{\frac{4}{3}\pi v^3 t^3} \,. \tag{53}$$

For the effective charge, the volume over which to integrate is given by the volume that can be reached within the retarded time:

$$\begin{aligned}
r' &\le R(t_r) && (54) \\
r' &\le v(t - r') && (55) \\
r' &\le \frac{vt}{1+v} && (56)
\end{aligned}$$

Hence, the effective charge is given by:

$$Q_{\text{eff}}(t) = \int_0^{R_{\text{eff}}} \rho(t_r) \cdot 4\pi r'^2 dr' \tag{57}$$

$$= \frac{3Q}{v^3} \int_0^{\frac{vt}{1+v}} \frac{r'^2}{(t-r')^3} dr' \tag{58}$$

Using $u = t - r'$, we get:

$$Q_{\text{eff}}(t) = \frac{3Q}{v^3} \int_{\frac{t}{1+v}}^{t} \frac{t^2 - 2tu + u^2}{u^3} du \tag{59}$$

$$= \frac{3Q}{v^3} \left[-\frac{t^2}{2u^2} + \frac{2t}{u} + \ln u \right]_{\frac{t}{1+v}}^{t} \tag{60}$$

$$= \frac{3Q}{v^3} \left[\left(-\frac{1}{2} + 2 + \ln t \right) - \left(-\frac{(1+v)^2}{2} + 2(1+v) + \ln\left(\frac{t}{1+v} \right) \right) \right] \tag{61}$$

$$= \frac{3Q}{v^3} \left[\ln(1+v) - v + \frac{v^2}{2} \right] \tag{62}$$

Using the Taylor expansion of the log for $v \ll 1$, we get:

$$Q_{\text{eff}} \approx \frac{3Q}{v^3} \left[\left(v - \frac{v^2}{2} + \frac{v^3}{3} - \frac{v^4}{4} \right) - v + \frac{v^2}{2} \right] \tag{63}$$

$$\approx Q \left(1 - \frac{3}{4} v \right) \tag{64}$$