Problem Set 6: Electromagnetic Waves IV

1 Picking Frequencies in Waveguides (5 points)

Consider a rectangular waveguide with dimensions 4 cm × 2 cm.

(a) What TE modes will propagate in this waveguide, if the driving frequency is 8 GHz?

> **SOLUTION:**
> For the rectangular waveguide, $a = 4\text{cm}$ and $b = 2\text{cm}$. The cutoff frequencies for the TE modes is given by:
>
> $$\omega_{mn} = \pi c \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \tag{1}$$
>
> The lowest frequency a propagating mode has is $\omega_{10} = \frac{\pi c}{a} = \frac{3.14*3*10^8}{.04}\text{rad}^{-1} = 23.5 \times 10^9 \text{rad}^{-1}$, which corresponds to $f_{10} = \frac{\omega_{10}}{2\pi} = 3.8\text{GHz}$. Noting that $a = 2b$, we see that the next cutoff frequency is:
>
> $$f_{01} = 2f_{10} = f_{20} = 7.2\text{GHz} \tag{2}$$
>
> The higher modes have cutoff larger than 8GHz. Hence, the modes that will propagate are TE_{10},TE_{01} and TE_{20}.

(b) Suppose you wanted to excite only one TE mode; what range of frequencies could you use? What are the corresponding wavelengths of an electromagnetic wave within that range of frequencies?

> **SOLUTION:**
> If only one mode had to be exited, the range of permissible frequencies is $3.8\text{GHz} \leq f < 7.2\text{GHz}$. The corresponding wavelengths are
>
> $$\frac{c}{7.2\text{GHz}} < \lambda \leq \frac{c}{3.8\text{GHz}} \tag{3}$$
>
> $$41.6\text{mm} < \lambda \leq 78.9\text{mm} \tag{4}$$

2 The Speed of Energy Transport is the Group Velocity (10 points)

Show that the energy in the TE_{mn} mode travels at the group velocity,

$$v_g = c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}. \tag{5}$$

To do this, find the time-averaged Poynting vector $\langle \mathbf{S} \rangle$ and the energy density $\langle u \rangle$, and integrate over the cross section of the wave guide to get the energy per unit time and per unit length carried by the wave, and then take their ratio.

> **SOLUTION:**
> The TE_{mn} modes have
>
> $$B_z = B_0 \cos(k_x x)\cos(k_y y) e^{i(kz - \omega t)} \quad \text{and} \quad E_z = 0\,, \tag{6}$$

with $k_x = \frac{m\pi}{a}$ and $k_y = \frac{n\pi}{b}$. The other components of the field can be obtained using Maxwell equations:

$$E_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\,\partial_x E_z + \omega\,\partial_y B_z\right) = \frac{-i}{(k_x^2+k_y^2)}\omega k_y B_0 \cos(k_x x)\sin(k_y y)e^{i(kz-\omega t)} \tag{7}$$

$$E_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\,\partial_y E_z - \omega\,\partial_x B_z\right) = \frac{i}{(k_x^2+k_y^2)}\omega k_x B_0 \sin(k_x x)\cos(k_y y)e^{i(kz-\omega t)} \tag{8}$$

$$B_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\,\partial_x B_z - \frac{\omega}{c^2}\,\partial_y E_z\right) = \frac{-i}{(k_x^2+k_y^2)}k k_x B_0 \sin(k_x x)\cos(k_y y)e^{i(kz-\omega t)} \tag{9}$$

$$B_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\,\partial_y B_z + \frac{\omega}{c^2}\,\partial_x E_z\right) = \frac{-i}{(k_x^2+k_y^2)}k k_y B_0 \cos(k_x x)\sin(k_y y)e^{i(kz-\omega t)} \tag{10}$$

$$\tag{11}$$

The time averaged component of the Poynting vector in the z-direction is given by:

$$\langle S_z \rangle = \frac{1}{2\mu_0}\mathrm{Re}(E_x B_y^* - E_y B_x^*) \tag{12}$$

$$= \frac{1}{2\mu_0}\frac{1}{(k_x^2+k_y^2)^2}\omega k B_0^2 (k_y^2 \cos^2(k_x x)\sin^2(k_y y) + k_x^2 \sin^2(k_x x)\cos^2(k_y y)) \tag{13}$$

$$\tag{14}$$

Noting that

$$\int_0^a \mathrm{d}x \cos^2\left(\frac{n\pi x}{a}\right) = \int_0^a \mathrm{d}x \sin^2\left(\frac{n\pi x}{a}\right) = \frac{a}{2}\,, \tag{15}$$

the the above quantity integrated over the cross section is:

$$P = \int \mathrm{d}x\mathrm{d}y\,\langle S_z \rangle \tag{16}$$

$$= \frac{ab\omega k B_0^2}{8\mu_0(k_x^2+k_y^2)} \tag{17}$$

The time averaged energy density is

$$\langle u \rangle = \frac{1}{4}\mathrm{Re}\left(\epsilon_0 \mathbf{E}\cdot\mathbf{E}^* + \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}^*\right) = \frac{1}{4\mu_0}\mathrm{Re}\left(\frac{1}{c^2}\mathbf{E}\cdot\mathbf{E}^* + \mathbf{B}\cdot\mathbf{B}^*\right) \tag{18}$$

$$\tag{19}$$

Again, we need to integrate it over the cross-section to get:

$$U = \int \mathrm{d}x\mathrm{d}y\,\langle u \rangle \tag{20}$$

$$= \frac{1}{4\mu_0}\left[\frac{B_0^2}{(k_x^2+k_y^2)^2}\left(\frac{\omega^2}{c^2}+k^2\right)(k_x^2+k_y^2)\frac{ab}{4} + B_0^2\frac{ab}{4}\right] \tag{21}$$

$$= \frac{B_0^2 ab}{16\mu_0(k_x^2+k_y^2)}\left[\left(\frac{\omega^2}{c^2}+k^2\right) + (k_x^2+k_y^2)\right] \tag{22}$$

$$= \frac{B_0^2 ab\omega^2}{8\mu_0 c^2(k_x^2+k_y^2)} \tag{23}$$

Now, the ratio is:

$$\frac{P}{U} = \frac{c^2 k}{\omega} = c^2\sqrt{1-\frac{k_x^2+k_y^2}{\omega^2}} = c^2\sqrt{1-\frac{\omega_{mn}^2}{\omega^2}} = v_g \tag{24}$$

3 TM Modes (15 points)

Work out the theory of TM modes for a rectangular waveguide, just as we did TE modes in class.

(a) Show that there are no TM_{01} or TM_{10} modes.

(b) Find the longitudinal electric field, $E_z(x, y, z, t)$.

(c) Find the cut-off frequencies ω_{mn}.

(d) Find the phase and group velocities of the TM_{mn} mode.

(e) Find the ratio of the lowest TM cutoff frequency to the lowest TE cutoff frequency, for a given waveguide.

SOLUTION:
We set $B_z = 0$ and solve

$$\nabla_T^2 E_z + \left(\frac{\omega^2}{c^2} - k^2\right) E_z = 0 \tag{25}$$

to find TM_{mn} modes. The procedure is the same as it was for the TE_{mn} modes, with the only difference being the boundary conditions imposed.
So, we start by using separation of variables: $E_{0,z} = X(x)Y(y)$ and then using the above equation to arrive at:

$$X(x) = A\sin(k_x x) + B\cos(k_x x) \tag{26}$$

$$Y(y) = C\sin(k_y y) + D\cos(k_y y) \tag{27}$$

with

$$\frac{\omega^2}{c^2} = k^2 - k_x^2 - k_y^2\,. \tag{28}$$

Now we impose boundary condition $E_\parallel = 0$, which implies that E_z must be set to zero everywhere at boundary. This can be done by setting:

$$B = D = 0 \quad \text{and} \quad k_x = m\pi/a, k_y = m\pi/b\,, \quad m, n \in \mathbb{Z}\,. \tag{29}$$

(b) With this, we get the final expression

$$E_z = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{i(kz-\omega t)}\,. \tag{30}$$

(a) From the above expression, we see that having either $m = 0$ or $n = 0$ sets $E_z = 0$. Hence, the first non-trivial mode is TM_{11}.
(c) The cut-off frequencies are:

$$\omega_{mn} = \frac{1}{c^2}\sqrt{k_x^2 + k_y^2} = \frac{\pi}{c}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \tag{31}$$

with the first cut-off frequency being $\omega_{11} = \frac{\pi}{c\sqrt{a^2+b^2}}$.
(d) The phase and group velocities are:

$$v_p = \frac{\omega}{k} = \frac{\omega}{\sqrt{\frac{\omega^2}{c^2} - (k_x^2 + k_y^2)}} = \frac{c}{\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}} \tag{32}$$

$$v_g = \frac{\partial\omega}{\partial k} = \frac{c^2 k}{\omega} = c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2} \tag{33}$$

(e) the lowest TM and TE cutoff frequency is :

$$\frac{\omega_{c\mathrm{TM}}}{\omega_{c\mathrm{TE}}} = \frac{a}{\sqrt{a^2+b^2}} \tag{34}$$

4 Coaxial Waveguide (10 points)

Consider a coaxial waveguide, with an inner conductor with a circular cross section of radius a, and an outer conductor with a cylindrical surface of radius b. The space between the conductors can be taken to be vacuum, and the conductors can be taken to be perfect conductors. An illustration of this is shown in Fig. 1.

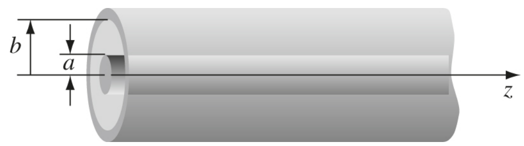

Figure 1: The coaxial waveguide.

(a) Show that the solutions (using cylindrical coordinates)

$$\begin{aligned} \mathbf{E}(s,\phi,z,t) &= \frac{A}{s}\cos(kz-\omega t)\hat{\mathbf{s}}\,, \\ \mathbf{B}(s,\phi,z,t) &= \frac{A}{cs}\cos(kz-\omega t)\hat{\boldsymbol{\phi}}\,, \end{aligned} \tag{35}$$

satisfy Maxwell's equations with $\omega/k = c$, and the boundary conditions $\mathbf{E}_\parallel = 0$ and $B_\perp = 0$. This is a TEM mode, with electric and magnetic fields, satisfying the usual dispersion relation in vacuum, $\omega/k = c$. Waves of any frequency can propagate in the coaxial waveguide.

SOLUTION:
The solutions above trivially satisfy $\mathbf{E}_\parallel = 0$ and $B_\perp = 0$, since $\hat{\mathbf{s}}$ is perpendicular to the boundary and $\hat{\phi}$ is parallel to the boundary. They also satisfy the Maxwell equations:

$$\nabla \cdot \mathbf{E} = \frac{1}{s}\partial_s(sE_s) = 0 \tag{36}$$

$$\nabla \cdot \mathbf{B} = \frac{1}{s}\partial_\phi B_\phi = 0 \tag{37}$$

$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = \partial_z E_s\,\hat{\phi} + \partial_t B_\phi\,\hat{\phi} = -\frac{A}{s}\left(k - \frac{\omega}{c}\right)\sin(kz-\omega t)\,\hat{\phi} = 0 \tag{38}$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\partial_t \mathbf{E} = -\partial_z B_\phi\,\hat{\mathbf{s}} - \frac{1}{c^2}\partial_t E_s\,\hat{\mathbf{s}} = \frac{A}{cs}\left(k - \frac{\omega}{c}\right)\sin(kz-\omega t)\,\hat{\mathbf{s}} = 0 \tag{39}$$

(b) Find the charge density $\lambda(z,t)$ and the current $I(z,t)$ on the inner conductor.

SOLUTION:
The charge density on the inner conductor is :

$$\sigma = (\epsilon_{\text{out}} E_\perp^{\text{out}} - \epsilon_{\text{in}} E_\perp^{\text{in}}) = \epsilon_0 E_s(a) = \frac{\epsilon_0 A}{a}\cos(kz-\omega t) \tag{40}$$

The linear surface charge density is

$$\lambda = \int_0^{2\pi} \mathrm{d}\phi\, a\sigma = 2\pi\epsilon_0 A\cos(kz-\omega t) \tag{41}$$

The surface current is given by:

$$\hat{\mathbf{n}} \times \mathbf{K} = \left(\frac{1}{\mu_{\text{out}}} \mathbf{B}_{\parallel}^{\text{out}} - \frac{1}{\mu_{\text{in}}} \mathbf{B}_{\parallel}^{\text{in}} \right) = \frac{1}{\mu_0} B_\phi(a) \, \hat{\phi} \tag{42}$$

$$\mathbf{K} = \hat{\mathbf{s}} \times \left(\frac{1}{\mu_0} B_\phi(a) \, \hat{\phi} \right) = \frac{A}{\mu_0 c a} \cos(kz - \omega t) \, \hat{\mathbf{z}} \, . \tag{43}$$

The total current is:

$$I = \int_0^{2\pi} \mathrm{d}\phi \, a K_z = \frac{2\pi A}{\mu_0 c} \cos(kz - \omega t) \tag{44}$$

5 A Fermi Problem: Refractive Index of Gas (10 points)

This is yet another Fermi problem! Once again, short of just looking up the values of the quantities directly, you can look up any data you would like about the systems that we're talking about to determine the quantities of interest described below. Feel free to use approximations and drop factors.

As we discussed in class, Cauchy's formula for the refractive index of a gas can be written as

$$n = 1 + A \left(1 + \frac{B}{\lambda^2} \right) . \tag{45}$$

At low frequencies $\omega \ll \omega_j$, where ω_j is a typical natural frequency of the electron bound to a nucleus, $n \approx 1 + A$. Let's consider hydrogen gas, made up of two hydrogen atoms bonded together covalently.

(a) Estimate the typical natural frequency ω_j of electrons bound to a nucleus, in accordance with the electron dipole model we discussed in class (*Hint:* Consider the hydrogen atom. What kind of potential does the electron sit in when bound to the proton? How far away is the electron from the proton?).

Considering the H-atom as a dipole, i.e., charge e separated by a Bohr radius, the natural frequency is:

$$\omega_j = \frac{E}{\hbar} = \frac{1}{\hbar} \frac{1}{4\pi\epsilon_0} \frac{e^2}{r_B} \approx \frac{1}{10^{-34}} \frac{1}{10^{-10}} \frac{10^{-38}}{10^{-11}} \text{Hz} = 10^{16} \text{Hz} \tag{46}$$

(b) Estimate what you expect the low frequency index of refraction to be, $n \approx 1 + A$, for hydrogen gas.

For Hydrogen gas, this would be

$$n \approx 1 + \frac{N e^2}{\epsilon_0 m \omega_j^2} \approx 1 + \frac{10^{25} \times 10^{-38}}{10^{-10} 10^{-30} 10^{32}} \approx 1 + 10^{-5} \tag{47}$$