Problem Set 5: Electromagnetic Waves III

1 Diamond (10 points)

The index of refraction of diamond is 2.42. For EM waves polarized in the plane of incidence, make a plot of $E_{0,T}/E_{0,I}$ and $E_{0,R}/E_{0,I}$ as functions of incidence angle θ_I for the air/diamond interface, going from air to diamond. In particular, calculate (a) the amplitudes at normal incidence, (b) Brewster's angle, and (c) the "crossover" angle at which the reflected and transmitted amplitudes are equal.

SOLUTION:
We shall use definitions

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-\left(\frac{1}{2.42}\right)^2\sin^2\theta_I}}{\cos\theta_I} \tag{1}$$

$$\beta = \frac{\mu_1 n_2}{\mu_2 n_1} \approx 2.42\,, \tag{2}$$

where we used Snell's law in the first expression. For waves polarized in the plane of incidence, the Fresnel's equations are

$$\frac{E_{0,R}}{E_{0,I}} = \frac{\alpha-\beta}{\alpha+\beta} \tag{3}$$

$$\frac{E_{0,T}}{E_{0,I}} = \frac{2}{\alpha+\beta} \tag{4}$$

This is plotted in the figure below.

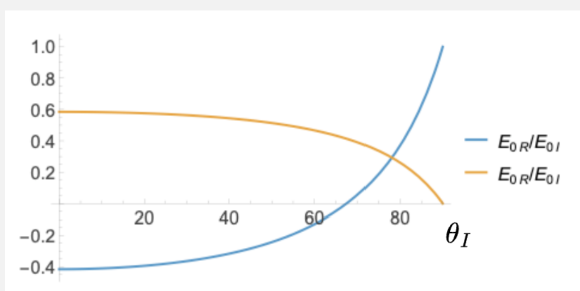

Figure 1

(a) At normal incidence, $\theta_I = 0$, $\alpha = 1$

$$\frac{E_{0,R}}{E_{0,I}} = \frac{1-2.42}{1+2.42} = -0.41 \quad \text{and} \quad \frac{E_{0,T}}{E_{0,I}} = \frac{2}{1+2.42} = 0.58 \tag{5}$$

(b) The Brewsters angle is given by setting $E_{0,R} = 0$, i.e. $\alpha = \beta$ which gives:

$$1-\left(\frac{1}{2.42}\right)^2\sin^2\theta_B = (2.42)^2\cos^2\theta_B \tag{6}$$

$$\sin^2\theta_B = \frac{(2.42)^2-1}{(2.42)^2-\frac{1}{(2.42)^2}} = 0.854 \tag{7}$$

$$\theta_B = 67.5^\circ \tag{8}$$

(c) The reflected and transmitted amplitudes are equal when:

$$\frac{\alpha-\beta}{\alpha+\beta} = \frac{2}{\alpha+\beta} \tag{9}$$

$$1-\left(\frac{1}{2.42}\right)^2 \sin^2\theta_C = (2+2.42)^2\cos^2\theta_C \tag{10}$$

$$\sin^2\theta_C = \frac{(2+2.42)^2-1}{(2+2.42)^2-\frac{1}{(2.42)^2}} = 0.854 \tag{11}$$

$$\theta_C = 78.1^\circ \tag{12}$$

2 Good Conductors (15 points)

(a) Show that the skin depth in a poor conductor ($\sigma \ll \omega\epsilon$) (with permittivity, permeability and conductivity given by ϵ, μ and σ) is

$$d = \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}\,. \tag{13}$$

SOLUTION:
The imaginary part of the wave vector within a conductor is given by:

$$\kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2} \tag{14}$$

In a poor conductor $\frac{\sigma}{\epsilon\omega} \ll 1$, and by expanding to first order in $\frac{\sigma}{\epsilon\omega}$, we get

$$\kappa \approx \omega\sqrt{\frac{\epsilon\mu}{2}}\left[1+\frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2-1\right]^{1/2} \tag{15}$$

$$= \omega\sqrt{\frac{\epsilon\mu}{2}}\frac{1}{\sqrt{2}}\frac{\sigma}{\epsilon\omega} \tag{16}$$

$$= \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \tag{17}$$

Taking the inverse of the above equation gives the skin depth.

(b) Show that the skin depth of a good conductor $\sigma \gg \omega\epsilon$ is

$$d = \frac{\lambda}{2\pi}\,, \tag{18}$$

where λ is the wavelength of the electromagnetic wave in the conductor.

SOLUTION:
When $\frac{\sigma}{\epsilon\omega} \gg 1$,

$$\kappa \approx \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\left(\frac{\sigma}{\epsilon\omega}\right)\right]^{1/2} \tag{19}$$

$$= \sqrt{\frac{\omega\mu\sigma}{2}} \tag{20}$$

Now, notice that the real part of the wavevector, given by

$$\kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2} \tag{21}$$

$$\approx \sqrt{\frac{\omega\mu\sigma}{2}}\,, \tag{22}$$

is the same as the imaginary part. Hence, the skin depth is

$$d = \frac{1}{\kappa} = \frac{1}{k} = \frac{2\pi}{\lambda}\,. \tag{23}$$

(c) Using the result in the previous part, Find the skin depth in nanometers for a typical metal ($\sigma \sim 10^7\,\Omega^{-1}\,\mathrm{m}^{-1}$) in the visible range ($\omega \sim 10^{15}\,\mathrm{s}^{-1}$, assuming $\epsilon = \epsilon_0$ and $\mu = \mu_0$. Why are metals opaque?

SOLUTION:
In a typical metal,

$$d = \frac{1}{\kappa} = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{24}$$

$$= \left(\sqrt{\frac{2}{(10^{15})(1.25 * 10^{-6})(10^7)}}\right)\mathrm{m} \tag{25}$$

$$\approx 10\,\mathrm{nm} \tag{26}$$

This skin depth suggests that visible light only goes in a few namometers into good conductors. This is the reason they are opaque.

(d) Show that in a good conductor, the magnetic field lags the electric field by $\pi/4$, and find an expression for the ratio of their amplitudes. For the typical numerical values in a metal given in the previous problem, what is this ratio?

SOLUTION:
The phase difference between the magnetic field and the electric field within the conductor is

$$\phi = \tan^{-1}\left(\frac{\kappa}{k}\right) = \tan^{-1}(1) = \frac{\pi}{4}\,. \tag{27}$$

The ratio of their amplitudes is

$$\frac{A_B}{A_E} = \sqrt{\epsilon\mu}\left[1+\left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{1/4} \tag{28}$$

$$\approx \sqrt{\epsilon\mu}\sqrt{\frac{\sigma}{\epsilon\omega}} \tag{29}$$

$$= \sqrt{\frac{\sigma\mu}{\omega}} \tag{30}$$

and within typical metals, this ratio is

$$\frac{A_B}{A_E} = \sqrt{\frac{10^7 * 1.25 * 10^{-6}}{10^{15}}}\,\mathrm{m}^{-1}\mathrm{s} \approx 10^{-7}\,\mathrm{m}^{-1}\mathrm{s}\,. \tag{31}$$

3 Energy Density and Intensity of EM Waves in a Conductor (5 points)

Consider a plane, monochromatic electromagnetic wave propagating along the z-axis inside a conductor, starting from $z = 0$, where the electric field has real amplitude given by E_0.

(a) Calculate the (time-averaged) energy density of an electromagnetic plane wave in a conducting medium. Show that the magnetic contribution always dominates.

SOLUTION:
The complex valued expression for this electromagnetic wave is

$$\mathbf{E} = E_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{\mathbf{x}} \tag{32}$$

$$\mathbf{B} = \frac{(k+i\kappa)}{\omega} E_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{\mathbf{y}} \tag{33}$$

Here, without loss of generality, we have assumed that the electric field is linearly polarized along the $\hat{\mathbf{x}}$ direction. Note, however, that the fields are the real part of above expressions. Hence, the energy density the electromagnetic wave is given by

$$u = \frac{\epsilon}{2}\mathrm{Re}[\mathbf{E}]^2 + \frac{1}{2\mu}\mathrm{Re}[\mathbf{B}]^2 \tag{34}$$

$$= \frac{\epsilon}{2} E_0^2 e^{-2\kappa z}\cos^2(kz-\omega t) + \frac{1}{2\mu}E_0^2 e^{-2\kappa z}\left(\frac{k}{\omega}\cos(kz-\omega t) - \frac{\kappa}{\omega}\sin(kz-\omega t)\right)^2 \tag{35}$$

Note that the time averages are

$$\langle\cos^2(kz-\omega t)\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{d}t\,\cos^2(kz-\omega t) = \frac{1}{2} \tag{36}$$

$$= \langle\sin^2(kz-\omega t)\rangle \tag{37}$$

$$\langle\cos(kz-\omega t)\sin(kz-\omega t)\rangle = 0 \tag{38}$$

Hence, the time averaged energy densities are

$$\langle u\rangle = \left(\frac{\epsilon}{4}E_0^2 e^{-2\kappa z}\right) + \left(\frac{1}{4\mu}\frac{k^2+\kappa^2}{\omega^2}E_0^2 e^{-2\kappa z}\right) \tag{39}$$

The ratio of the magnetic to the electric contribution to the time-averaged energy density is

$$\frac{\langle u_B\rangle}{\langle u_E\rangle} = \frac{k^2+\kappa^2}{\mu\epsilon\omega^2} \tag{40}$$

Recall that the wave equations in the conductor gives dispersion relations:

$$(k+i\kappa)^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \tag{41}$$

$$\implies k^2-\kappa^2 = \mu\epsilon\omega^2 \quad \text{and} \quad 2k\kappa = \mu\sigma\omega\,. \tag{42}$$

The above equations and now be used to obtain

$$k^2+\kappa^2 = \sqrt{(k^2-\kappa^2)^2 + (2k\kappa)^2} \tag{43}$$

$$= \sqrt{(\mu\epsilon\omega^2)^2 + (\mu\sigma\omega)^2}\,. \tag{44}$$

Substituting these back, we get

$$\frac{\langle u_B\rangle}{\langle u_E\rangle} = \sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} \tag{45}$$

which, for a good conductor is much greater than 1.

(b) Show that the intensity is as a function of z is

$$I = \frac{k}{2\mu\omega} E_0^2 e^{-2\kappa z} \,. \tag{46}$$

SOLUTION:
The Poynting vector for this wave is (recall from Problem Set 3)

$$\begin{aligned} \mathbf{S} &= \frac{1}{\mu}\mathrm{Re}\left[\frac{1}{2}\mathbf{E}\times\mathbf{B}^*\right] & (47)\\ &= \frac{1}{\mu}\mathrm{Re}\left[\frac{1}{2}\frac{k-i\kappa}{\omega}E_0^2 e^{-2\kappa z}\hat{\mathbf{z}}\right] & (48)\\ &= \frac{k}{2\mu\omega}E_0^2 e^{-2\kappa z}\,\hat{\mathbf{z}} & (49)\end{aligned}$$

The intensity is give by

$$I = |\langle \mathbf{S}\rangle .\, \hat{\mathbf{z}}| = \frac{k}{2\mu\omega}E_0^2 e^{-2\kappa z} \tag{50}$$

4 Evanescent Waves (20 points)

Let an electromagnetic wave be obliquely incident on an interface in the xy-plane at $z = 0$ between linear media 1 and 2 (not conductors). According to Snell's law, when light passes from an optically dense medium into a less dense one ($n_1 > n_2$), the propagation vector $\mathbf{k}$ bends away from the normal. In particular, if the light is incident at the **critical angle**

$$\theta_c \equiv \sin^{-1}\left(\frac{n_2}{n_1}\right) , \tag{51}$$

then $\theta_T = \pi/2$, and the transmitted wavevector just grazes the surface. If $\theta_I > \theta_c$, then there is no wave propagation in the z-direction, only a reflected one. This phenomenon is known as **total internal reflection**, and is the physical phenomenon underpinning how fiber optics works (and also periscopes!). But the electric and magnetic fields are not zero in medium 2; we get what is called an **evanescent wave**, which is rapidly attenuated and transport no energy into medium 2.

To get an expression for the evanescent wave, we can use the same expression for the transmitted electric and magnetic fields with oblique incidence,

$$\mathbf{E}_T(\boldsymbol{r},t) = \mathbf{E}_{0,T}e^{i(\mathbf{k}_T\cdot\boldsymbol{r}-\omega t)} , \qquad \mathbf{B}_T(\boldsymbol{r},t) = \frac{1}{v_2}\hat{\mathbf{k}}_T\times\mathbf{E}_T \,, \tag{52}$$

with $k_T = \omega n_2/c$, and

$$\mathbf{k}_T = k_T(\sin\theta_T\,\hat{\mathbf{x}} + \cos\theta_T\,\hat{\mathbf{z}}) \,. \tag{53}$$

By Snell's law,

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I \,, \tag{54}$$

where now $\sin\theta_T > 1$, and

$$\cos\theta_T = \sqrt{1-\sin^2\theta_T} = i\sqrt{\sin^2\theta_T - 1} \tag{55}$$

is *imaginary* (θ_T is obviously at this point no longer a physical angle).

(a) Show that

$$\mathbf{E}_T(\boldsymbol{r}, t) = \mathbf{E}_{0,T} e^{-\kappa z} e^{i(kx - \omega t)} \,, \tag{56}$$

where

$$\kappa \equiv \frac{\omega}{c}\sqrt{n_1^2 \sin^2\theta_I - n_2^2} \,, \qquad k \equiv \frac{\omega n_1}{c} \sin\theta_I \,. \tag{57}$$

This is a wave propagating in the x-direction, *parallel* to the interface, and attenuating in the z-direction.

SOLUTION:
As given above,

$$\begin{aligned}
\mathbf{k}_T &= k_T(\sin\theta_T \hat{\mathbf{x}} + \cos\theta_T \hat{\mathbf{z}}) && (58)\\
&= \frac{\omega n_2}{c}\left(\sin\theta_T \hat{\mathbf{x}} + i\sqrt{\sin^2\theta_T - 1}\,\hat{\mathbf{z}}\right) && (59)\\
&= \frac{\omega}{c}\left(n_2 \sin\theta_T \hat{\mathbf{x}} + i\sqrt{n_2^2\sin^2\theta_T - n_2^2}\,\hat{\mathbf{z}}\right) . && (60)
\end{aligned}$$

We can now use Snell's law to get

$$\begin{aligned}
\mathbf{k}_T &= \frac{\omega}{c}\left(n_1 \sin\theta_I \hat{\mathbf{x}} + i\sqrt{n_2^2\sin^2\theta_I - n_2^2}\,\hat{\mathbf{z}}\right) && (61)\\
&= k\,\hat{\mathbf{x}} + i\kappa\,\hat{\mathbf{z}} \cdot && (62)
\end{aligned}$$

Substituting $\mathbf{k}_T$ into the equation for $\mathbf{E}_T(\mathbf{r}, t)$, we arrive at the required expression.

(b) Noting that $\alpha \equiv \cos\theta_T / \cos\theta_I$ is now imaginary, calculate the reflection coefficient for polarization parallel to the plane of incidence.

SOLUTION:
Note that we now have:

$$\begin{aligned}
\alpha &= \frac{\cos\theta_T}{\cos\theta_I} = \frac{i\sqrt{\sin\theta_T - 1}}{\cos\theta_I} = \frac{i\sqrt{n_1^2\sin\theta_I - n_2^2}}{n_2\cos\theta_I} = i|\alpha| && (63)\\
\beta &= \frac{\mu_1 n_2}{\mu_2 n_1} && (64)
\end{aligned}$$

α is purely imaginary and β is real.
The Fresnel equations for polarization parallel to the plane of reflection give:

$$\begin{aligned}
\frac{E_{0,R}}{E_{0,I}} &= \frac{\alpha - \beta}{\alpha + \beta} && (65)\\
&= \frac{i|\alpha| - \beta}{i|\alpha| + \beta} && (66)
\end{aligned}$$

Taking the absolute value, we see that

$$\left|\frac{E_{0,R}}{E_{0,I}}\right| = 1 \,. \tag{67}$$

Hence, we can define $\phi = -2\tan^{-1}(|\alpha|/\beta)$ and assuming the plane of incidence is the xz-plane, the reflected electromagnetic waves can be written as:

$$\mathbf{E}_{0,R} = e^{i\phi} E_{0,I}\, e^{-i(\mathbf{k}_R \cdot \mathbf{r} + \omega t)} \hat{\mathbf{k}}_R \times \hat{\mathbf{y}} \tag{68}$$

$$\mathbf{B}_{0,R} = \frac{1}{v_1} \hat{\mathbf{k}}_R \times \mathbf{E}_{0,R} \tag{69}$$

$$= -e^{i\phi} \frac{E_{0,I}}{v_1}\, e^{-i(\mathbf{k}_R \cdot \mathbf{r} + \omega t)}\, \hat{\mathbf{y}} \tag{70}$$

The Poynting vector for the reflected wave and it corresponding intensity reflected at the interface is given by:

$$\mathbf{S} = \frac{1}{2\mu} \mathrm{Re}[\mathbf{E}_{0,R} \times \mathbf{B}^*_{0,R}] \tag{71}$$

$$= \frac{1}{2\mu} \frac{E^2_{0,I}}{v_1}\, \hat{\mathbf{k}}_R \tag{72}$$

$$I_R = |\langle \mathbf{S} \rangle . \hat{\mathbf{z}}| = \frac{1}{2\mu} \frac{E^2_{0,I}}{v_1} = I_I \tag{73}$$

The above expression shows that the intensity incident and reflected are the same, and hence the reflection coefficient is

$$R = \frac{I_R}{I_I} = 1\,, \tag{74}$$

when $\left|\frac{E_{0,R}}{E_{0,I}}\right| = 1$.

(c) Do the same for polarization perpendicular to the plane of incidence (which you should have worked out in your previous problem set). You should see from this and the previous part that total internal reflection is extremely useful for changing the direction of a wave with zero energy loss!

SOLUTION:
The Fresnel equations for polarization perpendicular to the plane of reflection give the reflection coefficient:

$$\frac{E_{0,R}}{E_{0,I}} = \frac{1 - \alpha\beta}{1 + \alpha\beta} \tag{75}$$

$$= \frac{1 - i|\alpha|\beta}{1 + i|\alpha|\beta} \tag{76}$$

Again, taking the absolute value, we see that

$$\left|\frac{E_{0,R}}{E_{0,I}}\right| = 1\,. \tag{77}$$

With similar arguments as in part(b), this implies that the reflection coefficient is 1.

(d) In the case of polarization perpendicular to the plane of incidence, show that the real evanescent fields are

$$\begin{aligned} \mathbf{E}(\boldsymbol{r}, t) &= E_0 e^{-\kappa z} \cos(kx - \omega t) \hat{\mathbf{y}}\,, \\ \mathbf{B}(\boldsymbol{r}, t) &= \frac{E_0}{\omega} e^{-\kappa z} [\kappa \sin(kx - \omega t) \hat{\mathbf{x}} + k \cos(kx - \omega t) \hat{\mathbf{z}}]\,. \end{aligned} \tag{78}$$

SOLUTION:
For perpendicular polarization, we can consider $\mathbf{E}_{0,I} = E_{0,I}\,\hat{\mathbf{y}}$ and Fresnel's equations gives us

$$E_{0,T} = \frac{2}{\alpha+\beta}E_{0,I} \tag{79}$$

$$= \frac{2}{i|\alpha|+\beta}E_{0,I} \tag{80}$$

Substituting this into the expression from part (a), we get:

$$\mathbf{E}_T(\mathbf{r},t) = \frac{2}{i|\alpha|+\beta}E_{0,I}\,e^{-\kappa z}e^{i(kx-\omega t)}\hat{\mathbf{y}} \tag{81}$$

Taking the real part of the above equation, we get

$$\mathbf{E}(\mathbf{r},t) = E_0\,e^{-\kappa z}\cos\left(kx-\omega t\right)\hat{\mathbf{y}} \quad \text{with} \quad E_0 = \frac{2}{\beta}E_{0,I}. \tag{82}$$

The magnetic field is give by:

$$\mathbf{B}_T(\mathbf{r},t) = \frac{1}{v_2}\hat{\mathbf{k}}_T \times \mathbf{E}_T \tag{83}$$

$$= \frac{1}{v_2}\frac{(k\,\hat{\mathbf{x}} + i\kappa\,\hat{\mathbf{z}})}{\sqrt{k^2-\kappa^2}} \times \left(E_0\,e^{-\kappa z}e^{i(kx-\omega t)}\,\hat{\mathbf{y}}\right) \tag{84}$$

$$= \frac{1}{v_2}\frac{(k\,\hat{\mathbf{z}} - i\kappa\,\hat{\mathbf{x}})}{\frac{\omega n_2}{c}}E_0\,e^{-\kappa z}e^{i(kx-\omega t)} \tag{85}$$

Since $v_2 = \frac{c}{n_2}$, the above expression simplifies to

$$\mathbf{B}_T(\mathbf{r},t) = \frac{E_0}{\omega}\,e^{-\kappa z}e^{i(kx-\omega t)}\,(k\,\hat{\mathbf{z}} - i\kappa\,\hat{\mathbf{x}}) \tag{86}$$

Taking the real part of the above expression, we get:

$$\mathbf{B}(\mathbf{r},t) = \frac{E_0}{\omega}\,e^{-\kappa z}\left[\cos(kx-\omega t)k\,\hat{\mathbf{z}} + \sin(kx-\omega t)\kappa\,\hat{\mathbf{x}}\right] \tag{87}$$

(e) Check that the fields above satisfy all of Maxwell's equations in linear media.

SOLUTION:

Maxwell equations in linear media

$$\nabla \cdot \mathbf{E} = \partial_y (E_0\, e^{-\kappa z} \cos(kx - \omega t)) = 0 \tag{88}$$

$$\nabla \cdot \mathbf{B} = \frac{E_0}{\omega}\, e^{-\kappa z}\, [-\kappa k \cos(kx - \omega t) + k\kappa \cos(kx - \omega t)] = 0 \tag{89}$$

$$\begin{aligned}\nabla \times \mathbf{E} + \partial_t \mathbf{B} &= E_0\, e^{-\kappa z}[-k \sin(kx - \omega t)\, \hat{\mathbf{z}} + \kappa \cos(kx - \omega t)\, \hat{\mathbf{x}}] \\ &\quad + \frac{E_0}{\omega}\, e^{-\kappa z}\, [\omega k \sin(kx - \omega t)\, \hat{\mathbf{z}} - \omega\kappa \cos(kx - \omega t)\, \hat{\mathbf{x}}] = 0 \end{aligned} \tag{90}$$

$$\begin{aligned}\nabla \times \mathbf{B} - \epsilon\mu \partial_t \mathbf{E} &= \frac{E_0}{\omega}\, e^{-\kappa z}[-\kappa^2 \sin(kx - \omega t) + k^2 \sin(kx - \omega t)]\hat{\mathbf{y}} \\ &\quad - \epsilon\mu\omega E_0\, e^{-\kappa z} \sin(kx - \omega t))\, \hat{\mathbf{y}} \end{aligned} \tag{91}$$

$$= \left(\frac{k^2 - \kappa^2}{\omega} - \frac{\omega}{v_2^2} \right) E_0\, e^{-\kappa z} \sin(kx - \omega t))\, \hat{\mathbf{y}} \tag{92}$$

$$= \left(\frac{\omega n_2^2}{c^2} - \frac{\omega}{v_2^2} \right) E_0\, e^{-\kappa z} \sin(kx - \omega t))\, \hat{\mathbf{y}} = 0 \tag{93}$$

(f) For the fields above, construct the Poynting vector, and show that, on average, no energy is transmitted in the $\hat{\mathbf{z}}$-direction.

SOLUTION:
The Poynting vector is:

$$\mathbf{S} = \frac{1}{\mu} \mathbf{E} \times \mathbf{B} \tag{94}$$

$$= \frac{1}{\mu} \frac{E_0^2}{\omega} e^{-2\kappa z} \cos(kx - \omega t)[-\kappa \sin(kx - \omega t)\hat{\mathbf{z}} + k \cos(kx - \omega t)\hat{\mathbf{x}}] \tag{95}$$

As in the previous problem, the time average of

$$\langle \cos(kz - \omega t) \sin(kz - \omega t) \rangle = 0. \tag{96}$$

Hence,

$$\langle \mathbf{S} \rangle . \hat{\mathbf{z}} = 0\,. \tag{97}$$