Problem Set 4: Electromagnetic Waves II

1 Three Media (20 points)

Consider the normal incidence of light traveling in the $+z$-direction on an interface at $z = 0$, followed by another interface at $z = d$, with three different media in total (label them 1, 2 and 3 in increasing z). Take all media to be linear and homogeneous, with $\mu_1 = \mu_2 = \mu_3 = \mu_0$, but with different refractive indices n_1, n_2 and n_3.

(a) For each medium, decide if there is a wave traveling in the $+z$-direction, and if there is a wave traveling in the $-z$-direction. Write down expressions for the electric and magnetic fields for each of these waves in the complex exponential form, with complex amplitudes.

SOLUTION:
For an incident wave along $+\hat{\mathbf{z}}$ in medium 1, there is transmission and reflection at $z = 0$ and at $z = d$, which implies waves traveling along $+\hat{\mathbf{z}}$ in medium 1, 2 and 3, and $-\hat{\mathbf{z}}$ in medium 1 and 2:

$$\mathbf{E}_1 = (E_I e^{(k_1 z - \omega t)} + E_R e^{-i(k_1 z + \omega t)})\,\hat{\mathbf{x}} \qquad \mathbf{B}_1 = \frac{1}{v_1}(E_I e^{i(k_1 z - \omega t)} - E_R e^{-i(k_1 z + \omega t)})\,\hat{\mathbf{y}} \tag{1}$$

$$\mathbf{E}_2 = (E_A e^{i(k_2 z - \omega t)} + E_B e^{-i(k_2 z + \omega t)})\,\hat{\mathbf{x}} \qquad \mathbf{B}_2 = \frac{1}{v_2}(E_A e^{i(k_2 z - \omega t)} - E_B e^{-i(k_2 z + \omega t)})\,\hat{\mathbf{y}} \tag{2}$$

$$\mathbf{E}_1 = E_T e^{i(k_3 z - \omega t)}\,\hat{\mathbf{x}} \qquad \mathbf{B}_1 = \frac{1}{v_3} E_T e^{i(k_3 z - \omega t)}\,\hat{\mathbf{y}} \tag{3}$$

$$\tag{4}$$

where $v_i = \frac{c}{n_i}$ and $k_i = \frac{\omega}{v_i}$.

(b) Using suitable boundary conditions at both $z = 0$ and $z = d$, find a relation between the wave amplitude transmitted into medium 3 with respect to the incident amplitude in medium 1. Show that the transmission coefficient is given by

$$T^{-1} = \frac{1}{4 n_1 n_3}\left[(n_1 + n_3)^2 + \frac{(n_1^2 - n_2^2)(n_3^2 - n_2^2)}{n_2^2}\sin^2(k_2 d)\right], \tag{5}$$

where k_2 is the wavenumber in medium 2.

SOLUTION:
The boundary conditions at $z = 0$ are:

$$E_I + E_R = E_A + E_B \tag{6}$$

$$\frac{1}{v_1}(E_I - E_R) = \frac{1}{v_2}(E_A - E_B) \tag{7}$$

and those at $z = d$ are:

$$E_A e^{ik_2 d} + E_B e^{-ik_2 d} = E_T e^{ik_3 d} \tag{8}$$

$$\frac{1}{v_2}(E_A e^{ik_2 d} - E_B e^{-ik_2 d}) = \frac{1}{v_3} E_T e^{ik_3 d} \tag{9}$$

In order to find the transmission coefficient, we solve the above equations to find E_T/E_I. From the first set of equations, we can eliminate E_R

$$E_I = \frac{v_1 + v_2}{2 v_2} E_A + \frac{v_2 - v_1}{2 v_2} E_B = \frac{n_1 + n_2}{2 n_1} E_A + \frac{n_1 - n_2}{2 n_1} E_B \tag{10}$$

Solving the second set, we can find

$$E_A = \frac{v_2 + v_3}{2v_3} E_T e^{i(k_3 - k_2)d} = \frac{n_2 + n_3}{2n_2} E_T e^{i(k_3 - k_2)d} \tag{11}$$

$$E_B = \frac{v_3 - v_2}{2v_3} E_T e^{i(k_3 + k_2)d} = \frac{n_2 - n_3}{2n_2} E_T e^{i(k_3 + k_2)d} \tag{12}$$

Using these relations in the above equation:

$$\frac{E_I}{E_T} = \frac{(n_1 + n_2)(n_2 + n_3)}{4n_1 n_2} e^{i(k_3 - k_2)d} + \frac{(n_1 - n_2)(n_2 - n_3)}{4n_1 n_2} e^{i(k_3 + k_2)d} \tag{13}$$

$$= \frac{e^{ik_3}}{4n_1 n_2} \left(2n_1(n_1 + n_3)\cos(k_2 d) + 2i(n_1 n_3 + n_2^2)\sin(k_2 d)\right) \tag{14}$$

The inverse of the transmission coefficient is given by (note that $\varepsilon_i = \frac{1}{v_i^2 \mu_0}$):

$$T^{-1} = \frac{v_1 \varepsilon_1 |E_I|^2}{v_3 \varepsilon_3 |E_T|^2} = \frac{v_3}{v_1} \left|\frac{E_I}{E_T}\right|^2 = \frac{n_1}{n_3} \left|\frac{E_I}{E_T}\right|^2 \tag{15}$$

$$= \frac{1}{4n_1^2 n_2^2} \left(n_1^2 (n_1 + n_3)^2 \cos^2(k_2 d) + (n_1 n_3 + n_2^2)^2 \sin(k_2 d)\right) \tag{16}$$

Using $\cos^2\theta = 1 - \sin^2\theta$ and simplifying, we arrive at the required expression for T^{-1}

(c) Suppose now you have an incident wave in medium 3, leading to a transmitted wave in medium 1. What is the transmission coefficient now? (*Hint:* make use of the result in the previous part, no need to reinvent the wheel!).

SOLUTION:
The transmission coefficient for this case can be obtained by swapping $n_1 \leftrightarrow n_3$ in the expression for the transmission coefficient. The expression in (b) is symmetric to this exchange, which implies that the transmission coefficient is the same in either direction.

(d) Light from an aquarium goes from water ($n = 4/3$) through a plane of glass ($n = 3/2$) into air ($n = 1$). Assuming a monochromatic plane wave and that it strikes the glass at normal incidence, find the minimum and maximum transmission coefficients. Will you have any problems looking into the aquarium?

SOLUTION:
We use the expression in (b) with $n_1 = 1$, $n_2 = 3/2$ and $n_3 = 4/3$. The minimum and maximum is simply given by the extrema of $\sin^2(k_2 d)$, i.e. setting it to 0 or 1.

$$T_{\max} = \frac{4n_1 n_3}{(n_1 + n_3)^2} = 0.98 \tag{17}$$

$$T_{\min} = \frac{4n_1 n_3}{(n_1 + n_3)^2 + \frac{(n_1^2 - n_2^2)(n_3^2 - n_2^2)}{n_2^2}} = 0.93 \tag{18}$$

Since the transmission coefficients are quite large, you will have no problems looking into the aquarium.

2 Oblique Incidence with Orthogonal Polarization (20 points)

In class, we discussed oblique incidence of an electromagnetic wave on an interface (separating linear medium 1 with μ_1, ϵ_1 from linear medium 2 with μ_2, ϵ_2) with polarization parallel to the plane of incidence. We'll now go through the case where the polarization of the incident wave is *orthogonal* to the plane of incidence.

(a) Draw a diagram of the interface on the xz-plane at $z = 0$, showing the electric and magnetic field orientations, the wavevectors, and the angle of incidence, reflection and transmission, θ_I, θ_R and θ_T respectively.

SOLUTION:

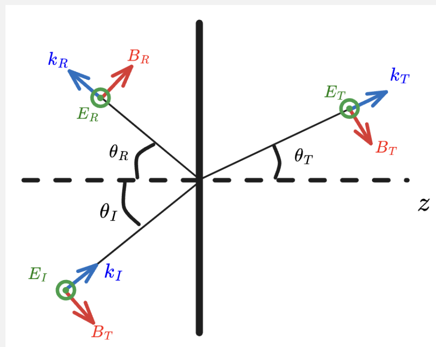

Figure 1

(b) Starting from the general boundary conditions, determine Fresnel's equations for this case, i.e. find the relation between the transmitted amplitude $E_{0,T}$ and the incident amplitude $E_{0,I}$, as well as between the reflected amplitude $E_{0,R}$ and $E_{0,I}$. You may use $\alpha \equiv \cos\theta_T / \cos\theta_I$ and $\beta \equiv \mu_1 v_1/(\mu_2 v_2)$.

SOLUTION:
We start with the expression for the electric and magnetic fields:

$$\mathbf{E}_I = E_{0I} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)} \hat{\mathbf{y}} \tag{19}$$

$$\mathbf{B}_I = \frac{1}{v_1} E_{0I} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)} \left(-\cos\theta_1 \hat{\mathbf{x}} + \sin\theta_1 \hat{\mathbf{z}}\right) \tag{20}$$

$$\mathbf{E}_R = E_{0R} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} \hat{\mathbf{y}} \tag{21}$$

$$\mathbf{B}_R = \frac{1}{v_1} E_{0R} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} \left(\cos\theta_1 \hat{\mathbf{x}} + \sin\theta_1 \hat{\mathbf{z}}\right) \tag{22}$$

$$\mathbf{E}_T = E_{0T} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)} \hat{\mathbf{y}} \tag{23}$$

$$\mathbf{B}_T = \frac{1}{v_2} E_{0T} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)} \left(-\cos\theta_2 \hat{\mathbf{x}} + \sin\theta_2 \hat{\mathbf{z}}\right) \tag{24}$$

The boundary conditions are:

$$\epsilon_1 E_{\perp 1} = \epsilon_2 E_{\perp 2} \tag{25}$$

$$B_{\perp 1} = B_{\perp 2} \tag{26}$$

$$E_{\parallel 1} = E_{\parallel 2} \tag{27}$$

$$\frac{1}{\mu_1} B_{\parallel 1} = \frac{1}{\mu_2} B_{\parallel 2} \tag{28}$$

Now, we can use the boundary conditions to find the relationship between the different amplitudes. From the third condition:

$$E_{0I} + E_{0R} = E_{0T} \tag{29}$$

Using this along with the second condition, we get expression of Snell's law at the interface:

$$\frac{1}{v_1} E_{0I} \sin\theta_1 + \frac{1}{v_1} E_{0R} \sin\theta_1 = \frac{1}{v_2} E_{0T} \sin\theta_2 \tag{30}$$

$$E_{0I} + E_{0R} = \left(\frac{v_1 \sin\theta_2}{v_2 \sin\theta_1}\right) E_{0T} \tag{31}$$

$$\implies \frac{\sin\theta_2}{\sin\theta_1} = \frac{v_2}{v_1} \tag{32}$$

In the above expressions, we have the fact that $\theta_R = \theta_I$
Using the fourth condition,

$$\frac{1}{\mu_1}\left[-\frac{1}{v_1} E_{0I} \cos\theta_1 + \frac{1}{v_1} E_{0R} \cos\theta_1\right] = -\frac{1}{\mu_2 v_2} E_{0T} \cos\theta_2 \tag{33}$$

Simplifying the above expression with the given definitions of α and β,

$$E_{0I} - E_{0R} = \alpha\beta E_{0T} \tag{34}$$

Solving for the required expressions of $E_{0T}$and E_{0R} in terms ofE_{0I}

$$E_{0T} = \left(\frac{2}{1+\alpha\beta}\right) E_{0I} \tag{35}$$

$$E_{0R} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right) E_{0I} \tag{36}$$

(c) Sketch $E_{0,R}/E_{0,I}$ and $E_{0,T}/E_{0,I}$ as a function of θ_I, taking for concreteness $\mu_1 = \mu_2 = \mu_0$, and $\beta = n_2/n_1 = 1.5$.

SOLUTION:
For the given values,

$$\alpha = \frac{\cos\theta_2}{\cos\theta_1} = \frac{\sqrt{1-\sin^2\theta_2}}{\cos\theta_1} = \frac{\sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta}}{\cos\theta_1} \tag{37}$$

$$\alpha\beta = \frac{\sqrt{\beta^2 - \sin^2\theta_1}}{\cos\theta_1} = \frac{\sqrt{2.25-\sin^2\theta_1}}{\cos\theta_1} \tag{38}$$

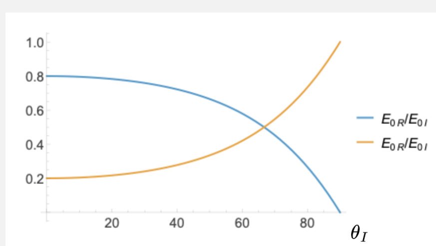

Figure 2

(d) For $\mu_1 = \mu_2 = \mu_0$, show that there is no Brewster's angle, i.e. $E_{0,R} \neq 0$ for any n_1 and n_2 other than $n_1 = n_2$ (this is the trivial case where the two media are indistinguishable from the point of view of EM waves, and so there is no reflection).

SOLUTION:
For $E_{0R} = 0$, we require $\alpha\beta = 1$. Using the expression from part(c), we see that this condition implies:

$$\alpha\beta = \frac{\sqrt{\beta^2 - \sin^2\theta_1}}{\cos\theta_1} = 1 \tag{39}$$

$$\implies \beta^2 = \sin^2\theta_1 + \cos^2\theta_1 = 1 \quad \text{or} \quad \beta = 1 \quad \text{or} \quad n_1 = n_2\,. \tag{40}$$

(e) Check that the transmission and reflection coefficients T and R satisfy $T + R = 1$.

SOLUTION:
The reflection and transmission coefficients are:

$$R = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2 \tag{41}$$

$$T = \frac{\epsilon_2 v_2 \cos\theta_2}{\epsilon_1 v_1 \cos\theta_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \frac{4\alpha\beta}{(1+\alpha\beta)^2} \tag{42}$$

Their sum gives:

$$T + R = \frac{(1-\alpha\beta)^2 + 4\alpha\beta}{(1+\alpha\beta)^2} = \frac{(1+\alpha\beta)^2}{(1+\alpha\beta)^2} = 1 \tag{43}$$

$$\tag{44}$$

(f) Check that for $\theta_I = 0$, you recover T and R for normal incidence.

SOLUTION:
If $\theta_I = 0$, then $\theta_T = 0$ and $\alpha = 1$ and we are left with

$$R = \frac{(1-\beta)^2}{(1+\beta)^2} = \frac{(n_1 - n_2)^2}{(n_1+n_2)^2} \tag{45}$$

$$T = \frac{4\beta}{(1+\beta)^2} = \frac{4n_1 n_2}{(n_1+n_2)^2} \tag{46}$$

These are the expression we found for normal incidence.

3 Radiation Pressure from the Sun vs. Gravity: A Fermi Problem (15 points)

The sun is emitting electromagnetic waves, and so is exerting an electromagnetic force radially outward. At the same time, it also has a gravitational pull acting in the opposite direction.

(a) Does the distance from the Sun of the object matter to determining which force is stronger? Why or why not?

SOLUTION:
The sun radiates spherically outward. Hence, the radiation flux goes down as $1/r^2$ and so the radiation pressure also drops as $1/r^2$. This is the same as drop of the gravitational force of an object from the sun. Hence, the distance doesn't matter in determining which force is stronger.

(b) Consider an object of characteristic size a with some density ρ. Under what conditions are the two forces balanced?

SOLUTION:
For an object of density ρ with radius a at a distance R from the sun, the gravitational force is:

$$F_G = \frac{GM_0}{R^2}\left(\rho\frac{4\pi a^3}{3}\right) \tag{47}$$

where M_0 is the mass of the sun. Assuming solar luminosity L_0, the intensity of radiation from the sun at a distance R is

$$I = \frac{L_0}{4\pi R^2} \tag{48}$$

The radiation pressure is (assuming the object is perfectly absorbing)

$$p = \frac{I}{c} \tag{49}$$

Hence the radiation force on the object is

$$F_{EM} = p(4\pi a^2) = \frac{L_0}{c}\frac{a^2}{R^2} \tag{50}$$

The two forces are balanced when

$$\frac{GM_0}{R^2}\left(\rho\frac{4\pi a^3}{3}\right) = \frac{L_0}{c}\frac{a^2}{R^2} \tag{51}$$

$$\text{or} \quad a\rho = \frac{3L_0}{4\pi c\, GM_0} \tag{52}$$

(c) For typical objects, what value of a would be required for the two forces to be balanced? Would bigger objects be more affected by radiation pressure, or gravitational pull?

SOLUTION:
Substituting typical values for the constants in the above expression:

$$a(10^3\text{kg/m}^3) = \frac{3(3.8*10^{26}\text{W})}{4(3.14)(3*10^8\text{m/s})(6.7*10^{-11}\text{Nm}^2/\text{kg}^2)(2*10^{30}\text{kg})} \tag{53}$$

$$a \approx 10^{-6}\text{m} \tag{54}$$

This gives radii of the order of microns, which would imply that this could hold for suspended dust particles in space. Clearly, bigger objects are affected more by the gravitational pull rather than the electromagnetic radiation force, since it scales cubically as opposed to quadratically with a.

NOTE: This is a Fermi problem. It is not a typical homework problem, but much more reminiscent of a problem you might encounter as a physicist. You're free to use any information you want, define any variable you want, but please don't plug in numbers until right at the end, so that the dependencies on various parameters are obvious. Estimates are highly encouraged—you should not be doing integrals or worrying about the exact geometry of things.