Problem Set 3: Electromagnetic Waves I

1 The Wave Equation (10 points)

(a) Consider the 3D wave equation

$$\nabla^2 \mathbf{\Psi}(\mathbf{r}, t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\mathbf{\Psi}(\mathbf{r}, t) = 0\,, \tag{1}$$

where $\mathbf{\Psi}(\boldsymbol{r}, t)$ is 3D-vector field at 3D position $\boldsymbol{r}$ and time t. Show that

$$\mathbf{\Psi}(\mathbf{r}, t) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\tilde{\mathbf{A}}(\mathbf{k})e^{i(\mathbf{k}\cdot\boldsymbol{r} - \omega t)} \tag{2}$$

is a solution to the wave equation, so long as $v = \omega/k$. This shows that any solution to the wave equation can be thought of as a linear combination of (uncountably infinitely) many sinusoidal plane waves (don't worry about the factor of $(2\pi)^3$, it's just a convention used in physics).

SOLUTION: We verify that the above expression is a solution by substituting $\mathbf{\Psi}(\mathbf{r}, t)$ in the wave equation.

$$\left(\partial_i\partial^i - \frac{1}{v^2}\partial_t^2\right)\mathbf{\Psi}(\mathbf{r}, t) = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^3}\tilde{\mathbf{A}}(\mathbf{k})\left(\partial_i\partial^i - \frac{1}{v^2}\partial_t^2\right)e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{3}$$

$$= \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^3}\tilde{\mathbf{A}}(\mathbf{k})\left(k^2 - \frac{1}{v^2}\omega^2\right)e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} = 0 \tag{4}$$

(b) Now consider the 3D wave equation

$$\nabla^2 \psi(\boldsymbol{r}, t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi(\boldsymbol{r}, t) = 0\,, \tag{5}$$

where $\psi(\boldsymbol{r}, t)$ is now a scalar field. Show that

$$\psi(r, t) = \frac{1}{r}F(r - vt) + \frac{1}{r}G(r + vt) \tag{6}$$

is a solution, nwhere $r = |\boldsymbol{r}|$, and F and G are functions of $r - vt$ and $r + vt$ only, respectively. What kind of symmetry do the solutions F and G have, and in what direction are they traveling?

SOLUTION:
In this case, it is simpler to use spherical coordinates. For simplicity, let's first consider $\psi_1(\mathbf{r}, t) =$

$\frac{1}{r}F(r-vt)$.

$$\nabla^2\psi_1(\mathbf{r},t) = \frac{1}{r^2}\partial_r(r^2\partial_r\psi(\mathbf{r},t)) \tag{7}$$
$$= \frac{1}{r^2}\partial_r\left(r^2\left(-\frac{1}{r^2}F(r-vt)+\frac{1}{r}F'(r-vt)\right)\right) \tag{8}$$
$$= \frac{1}{r^2}\partial_r\left(-F(r-vt)+rF'(r-vt)\right) \tag{9}$$
$$= \frac{1}{r}F''(r-vt) \tag{10}$$
$$\partial_t^2\psi_1(\mathbf{r},t) = \frac{v^2}{r}F''(r-vt) \tag{11}$$
$$\implies \nabla^2\psi_1(\mathbf{r},t) - \frac{1}{v^2}\partial_t^2\psi(\mathbf{r},t) = 0 \tag{12}$$

The math follows similarly for $\psi_2(\mathbf{r},t) = \frac{1}{r}G(r+vt)$. Both F and G have spherical symmetry about the origin $r=0$, since they are independent of ϕ and θ. $F(r-vt)$ travels radially outwards, and this can be understood by considering when the phase is constant; $r - vt = (\text{const}) \implies r = (\text{const}) + vt$. With a similar logic, $G(r+vt)$ travels radially inwards.

2 The Inhomogeneous Wave Equation (5 points)

Starting from Maxwell's equations *with* charges and currents, show that the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ satisfy the *inhomogeneous* wave equations,

$$\begin{aligned}\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} &= \frac{1}{\epsilon_0}\nabla\rho + \mu_0\frac{\partial\mathbf{J}}{\partial t}\,,\\ \nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} &= -\mu_0\nabla\times\mathbf{J}\,.\end{aligned} \tag{13}$$

SOLUTION:
We will use the following identity, which holds for any vector field $\mathbf{A}$:

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \tag{14}$$

Consider the curl of Faraday's law:

$$\nabla\times(\nabla\times\mathbf{E}+\partial_t\mathbf{B}) = 0 \tag{15}$$
$$\nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} + \partial_t(\nabla\times\mathbf{B}) = 0 \tag{16}$$

Now using Gauss law and the Ampere-Maxwell equation, we get:

$$\frac{1}{\epsilon_0}\nabla\rho - \nabla^2\mathbf{E} + \mu_0\epsilon_0\partial_t^2\mathbf{E} - \mu_0\partial_t\mathbf{J} = 0 \tag{17}$$

Rearranging and noting that $\epsilon_0\mu_0 = \frac{1}{c^2}$, we arrive at the wave equation for $\mathbf{E}$.
Similarly, consider the curl of the Ampere-Maxwell equation:

$$\nabla\times(\nabla\times\mathbf{B} - \mu_0\epsilon_0\partial_t\mathbf{E} - \mu_0\mathbf{J}) = 0 \tag{18}$$
$$\nabla(\nabla\cdot\mathbf{B}) - \nabla^2\mathbf{B} - \frac{1}{c^2}\partial_t(\nabla\times\mathbf{E}) - \mu_0\nabla\times\mathbf{J} = 0 \tag{19}$$

Using Faraday's law and the Gauss law for magnetism, we get:

$$-\nabla^2\mathbf{B} + \frac{1}{c^2}\partial_t^2\mathbf{B} - \mu_0\nabla\times\mathbf{J} = 0 \tag{20}$$

3 Products with Complex Exponentials (5 points)

I mentioned in class that one needs to be careful when multiplying two sinusoidal waves together as complex exponentials, since the unphysical imaginary parts become a real contribution. Nevertheless, exponentials are so nice that it would be a shame to let this problem stop us from using them. In that spirit, here are some tricks and related abuse of notation that you'll see very commonly in the literature:

(a) Let $f(\boldsymbol{r},t) = A\cos(\mathbf{k}_1 \cdot \boldsymbol{r} - \omega_1 t + \delta_1)$ and $g(\boldsymbol{r},t) = B\cos(\mathbf{k}_2 \cdot \boldsymbol{r} - \omega_2 t + \delta_2)$. Show that

$$f \cdot g = \mathrm{Re}\left[\frac{1}{2}(F \cdot G + F \cdot G^*)\right] , \tag{21}$$

where $F = A\exp[i(\mathbf{k}_1 \cdot \boldsymbol{r} - \omega_1 t + \delta_1)]$ is the complex exponential form of f, and likewise for G. In other words, you can take the product $f \cdot g$ using complex exponentials *not* by taking $F \cdot G$, but using the formula above instead.

SOLUTION:
With $\phi_1 = \mathbf{k}_1 \cdot \mathbf{r} - \omega_1 t + \delta_1$ and $\phi_2 = \mathbf{k}_2 \cdot \mathbf{r} - \omega_2 t + \delta_2$,

$$\mathrm{Re}\left[\frac{1}{2}(F \cdot (G + G^*))\right] = \mathrm{Re}\left[\frac{1}{2}(e^{i\phi_1} \cdot (2\cos\phi_2))\right] = \cos\phi_1 \cos\phi_2 = f \cdot g. \tag{22}$$

(b) Let $f(z,t) = A\cos(\omega t - \alpha)$ and $g(z,t) = B\cos(\omega t - \beta)$. Show that the time-average $\langle f \cdot g \rangle$ is given by

$$\langle f \cdot g \rangle = \frac{1}{2} F \cdot G^* = \frac{1}{2} F^* \cdot G \, , \tag{23}$$

where the time-average is defined as

$$\langle f \cdot g \rangle = \frac{1}{T}\int_0^T dt \, f \cdot g \, , \tag{24}$$

where $T \equiv 2\pi/\omega$ is the period of one oscillation, and F, G are once again the complex exponential form of f and g. Note that the RHS of Eq. (23) is a complex number, and to obtain the physical value of $\langle f \cdot g \rangle$, you must take the real part. This notation is standard in the literature, and you will not be reminded of this out in the wild!

SOLUTION:
Using the identity in part (a) in the above equation, we get:

$$\begin{aligned}
\langle f \cdot g \rangle &= \frac{1}{2T}\mathrm{Re}\left[\int_0^T dt \ (F \cdot G + F.G^*)\right] && (25)\\
&= \frac{AB}{2T}\mathrm{Re}\left[\int_0^T dt \ \left(e^{i(2\omega t - \alpha - \beta)} + e^{i(\beta - \alpha)}\right)\right] && (26)\\
&= \frac{AB}{2T}\mathrm{Re}\left[0 + Te^{i(\beta-\alpha)}\right] = \frac{AB}{2}\cos(\beta - \alpha) && (27)\\
&= \frac{1}{2}F \cdot G^* = \frac{1}{2}F^* \cdot G && (28)
\end{aligned}$$

(c) Do these results also work for vector fields $\mathbf{f}$ and $\mathbf{g}$, e.g. does

$$\langle \mathbf{f} \times \mathbf{g} \rangle \stackrel{?}{=} \frac{1}{2}\mathbf{F} \times \mathbf{G}^* \tag{29}$$

hold? Again, remember that the RHS is complex, and to get the physically sensible answer, you have to take the real part (*Hint:* Think in index notation.)

SOLUTION:
Yes, they do work for vector fields. In index notation, the above equation can be rewritten as:

$$\varepsilon^{ijk}\langle f_j \cdot g_k\rangle = \frac{1}{2}\varepsilon^{ijk}\ F_j \cdot G_k^* \tag{30}$$

This relation can be verified with a simple argument: since each component of a vector is a scalar, we can apply the identity in part(b) to obtain

$$\langle f_j \cdot g_k\rangle = \frac{1}{2}\ F_j \cdot G_k^* \tag{31}$$

for all j, k.

4 Energy and Momentum in Electromagnetic Waves (10 points)

Consider the monochromatic plane wave given by

$$\mathbf{E}(\boldsymbol{r},t) = E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{n}} \tag{32}$$

$$\mathbf{B}(\boldsymbol{r},t) = \frac{E_0}{c} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{k}}\times\hat{\mathbf{n}}\,. \tag{33}$$

(a) Using the results in the previous problem, evaluate the energy density $u(\boldsymbol{r},t)$, the momentum density $\wp(\boldsymbol{r},t)$, and the Poynting vector $\mathbf{S}(\mathbf{r},t)$ associated with this plane wave. Write down the relationship between u and $\mathbf{S}$, u and $\wp$, and $\mathbf{S}$ and $\wp$.

SOLUTION:
Using the relations from the previous problem, the energy density can be expressed as:

$$\begin{aligned} u(\mathbf{r},t) &= \frac{\epsilon_0}{2}\left(\frac{1}{2}\mathrm{Re}[\mathbf{E}\cdot(\mathbf{E}+\mathbf{E}^*)]\right) + \frac{1}{2\mu_0}\left(\frac{1}{2}\mathrm{Re}[\mathbf{B}\cdot(\mathbf{B}+\mathbf{B}^*)]\right) && (34)\\ &= \left(\frac{\epsilon_0 E_0^2}{2} + \frac{1}{2\mu_0}\frac{E_0^2}{c^2}\right)\ \frac{1}{2}\mathrm{Re}[e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(2\cos(\mathbf{k}\cdot\mathbf{r}-\omega t))] && (35)\\ &= \epsilon_0 E_0^2\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t) && (36)\end{aligned}$$

Similarly, the Poynting vector is:

$$\begin{aligned} \mathbf{S}(\mathbf{r},t) &= \frac{1}{\mu_0}\frac{1}{2}\mathrm{Re}\left[\mathbf{E}\times(\mathbf{B}+\mathbf{B}^*)\right] && (37)\\ &= \frac{E_0^2}{\mu_0 c}(\hat{\mathbf{n}}\times(\hat{\mathbf{k}}\times\hat{\mathbf{n}}))\left(\frac{1}{2}\mathrm{Re}[e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(2\cos(\mathbf{k}\cdot\mathbf{r}-\omega t))]\right) && (38)\\ &= \frac{E_0^2}{\mu_0 c}\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t)\,\hat{\mathbf{k}} && (39)\\ &= c\,u(\mathbf{r},t)\hat{\mathbf{k}} && (40)\end{aligned}$$

The momentum density is:

$$\begin{aligned} \wp &= \mu_0\epsilon_0\mathbf{S} = \frac{\epsilon_0 E_0^2}{c}\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t)\,\hat{\mathbf{k}} && (41)\\ &= \frac{1}{c}u(\mathbf{r},t)\,\hat{\mathbf{k}} && (42)\end{aligned}$$

(b) Using the results in the previous problem, evaluate the time-averaged quantities $\langle u\rangle$, $\langle\wp\rangle$, and $\langle\mathbf{S}\rangle$.

SOLUTION:
The time average is:

$$\langle u\rangle = \frac{\epsilon_0}{4}\mathrm{Re}\left[\mathbf{E}\cdot\mathbf{E}^*\right] + \frac{1}{4\mu_0}\mathrm{Re}\left[\mathbf{B}\cdot\mathbf{B}^*\right] \tag{43}$$

$$= \left(\frac{\epsilon_0 E_0^2}{2} + \frac{1}{2\mu_0}\frac{E_0^2}{c^2}\right)\frac{1}{2}\left(e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\cdot e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right) \tag{44}$$

$$= \frac{\epsilon_0 E_0^2}{2} \tag{45}$$

Similarly we obtain:

$$\langle\mathbf{S}\rangle = \frac{1}{2\mu_0}\mathrm{Re}[\mathbf{E}\times\mathbf{B}^*] \tag{46}$$

$$= \frac{c\epsilon_0 E_0^2}{2}\hat{\mathbf{k}} \tag{47}$$

$$\langle\wp\rangle = \frac{\epsilon_0 E_0^2}{2c}\hat{\mathbf{k}} \tag{48}$$

5 Polarization and Poynting Vectors (15 points)

Four electromagnetic waves are represented by the following expressions, in which the E^0 coefficients are all real, and the phase α is written explicitly:

$$\text{Wave 1: } \mathbf{E}_1 = E_1^0\hat{\mathbf{x}}e^{i(kz-\omega t)}\,, \qquad \mathbf{B}_1 = \frac{E_1^0}{c}\hat{\mathbf{y}}e^{i(kz-\omega t)}\,. \tag{49}$$

$$\text{Wave 2: } \mathbf{E}_2 = E_2^0\hat{\mathbf{y}}e^{i(kz-\omega t+\alpha)}\,, \qquad \mathbf{B}_2 = -\frac{E_2^0}{c}\hat{\mathbf{x}}e^{i(kz-\omega t+\alpha)}\,. \tag{50}$$

$$\text{Wave 3: } \mathbf{E}_3 = E_3^0\hat{\mathbf{x}}e^{i(kz-\omega t+\alpha)}\,, \qquad \mathbf{B}_3 = \frac{E_3^0}{c}\hat{\mathbf{y}}e^{i(kz-\omega t+\alpha)}\,. \tag{51}$$

$$\text{Wave 4: } \mathbf{E}_4 = E_1^0\hat{\mathbf{x}}e^{i(-kz-\omega t)}\,, \qquad \mathbf{B}_4 = -\frac{E_1^0}{c}\hat{\mathbf{y}}e^{i(-kz-\omega t)}\,. \tag{52}$$

You may find the previous problem very helpful.

(a) Calculate the time-averaged Poynting vector $\langle\mathbf{S}\rangle$ for the superposition of Wave 1 and Wave 2. Show that this quantity is just the sum of $\langle\mathbf{S}\rangle$ for the waves taken separately. Why? Describe the polarization when $\alpha = 0$. What happens when $E_2^0 = E_1^0$ and $\alpha = \pi/2$?

SOLUTION:
Using the results from the previous problem, we read off the time averaged Poynting vectors for Wave 1 and Wave 2 taken separately:

$$\langle\mathbf{S}_1\rangle = \frac{c\epsilon_0}{2}(E_1^0)^2\,\hat{\mathbf{z}} \tag{53}$$

$$\langle\mathbf{S}_2\rangle = -\frac{c\epsilon_0}{2}(E_2^0)^2\,\hat{\mathbf{z}} \tag{54}$$

$$\tag{55}$$

For the superposition, we have:

$$\mathbf{E}_{12} = \left(E_1^0\,\hat{\mathbf{x}} + E_2^0 e^{i\alpha}\,\hat{\mathbf{y}}\right)\, e^{i(kz-\omega t)}\,, \qquad \mathbf{B}_{12} = \frac{1}{c}\left(E_1^0\,\hat{\mathbf{x}} - E_2^0 e^{i\alpha}\,\hat{\mathbf{y}}\right)\, e^{i(kz-\omega t)} \tag{56}$$

$$\langle \mathbf{S}_{12}\rangle = \frac{1}{2\mu_0}\mathrm{Re}[\mathbf{E}_{12}\times\mathbf{B}_{12}^*] \tag{57}$$

$$= \frac{c\epsilon_0}{2}\left((E_1^0)^2\,\hat{\mathbf{z}} + (E_2^0)^2\,\hat{\mathbf{z}}\right) \tag{58}$$

$$= \langle \mathbf{S}_1\rangle + \langle \mathbf{S}_2\rangle \tag{59}$$

In this case, we get the sum of the quantity for the waves taken separately due to the cross terms $\mathbf{E}_1\times\mathbf{B}_2$ and $\mathbf{E}_2\times\mathbf{B}_1$ being zero.
When $\alpha = 0$, the electric field points along the direction $E_1^0\,\hat{\mathbf{x}} + E_2^0\,\hat{\mathbf{y}}$ and its magnitude is varying with time. This is a linearly polarized wave. When $\alpha = \frac{\pi}{2}$ and $E_1^0 = E_2^0 = E^0$, notice that the electric field,

$$\mathrm{Re}[\mathbf{E}_{12}] = E^0(\cos(kz-\omega t)\,\hat{\mathbf{x}} + \sin(kz-\omega t)\,\hat{\mathbf{y}}) \tag{60}$$

has a constant magnitude with a direction that rotates along the $x-y$ plane. This is a circularly polarized wave.

(b) Calculate $\langle\mathbf{S}\rangle$ for the superposition of Wave 1 and Wave 3. How is this different than the previous part?

SOLUTION:
For the superposition of Wave 1 and Wave 3:

$$\mathbf{E}_{13} = \left(E_1^0 + E_3^0 e^{i\alpha}\right) e^{i(kz-\omega t)}\,\hat{\mathbf{x}}\,, \qquad \mathbf{B}_{12} = \frac{1}{c}\left(E_1^0 + E_3^0 e^{i\alpha}\right)\hat{\mathbf{y}}\, e^{i(kz-\omega t)} \tag{61}$$

$$\langle \mathbf{S}_{13}\rangle = \frac{1}{2\mu_0}\mathrm{Re}[\mathbf{E}_{13}\times\mathbf{B}_{13}^*] \tag{62}$$

$$= \frac{c\epsilon_0}{2}\left((E_1^0)^2 + (E_2^0)^3 + 2E_1^0E_3^0\cos\alpha\right)\hat{\mathbf{z}} \tag{63}$$

Here, we see that we get the cross term that is dependent on the phase difference α.

(c) Calculate $\langle\mathbf{S}\rangle$ for the superposition of Wave 1 and Wave 4. Interpret the result.

SOLUTION:
For the superposition of Wave 1 and Wave 4:

$$\mathbf{E}_{14} = 2E_1^0\cos(kz)\, e^{-i\omega t}\,\hat{\mathbf{x}}\,, \qquad \mathbf{B}_{14} = \frac{1}{c}2E_1^0 i\sin(kz)\, e^{-i\omega t}\,\hat{\mathbf{y}} \tag{64}$$

$$\langle \mathbf{S}_{14}\rangle = \frac{1}{2\mu_0}\mathrm{Re}[\mathbf{E}_{14}\times\mathbf{B}_{14}^*] \tag{65}$$

$$= 0 \tag{66}$$

In this case, the Poynting vector is zero because we have a standing wave, formed by superposition of two identical waves traveling in opposite directions and there is no net energy flow.

(d) For the superposition of Wave 1 and Wave 4, calculate the *time-dependent* energy density $u(t)$, keeping track of the electric field and magnetic field portions separately; show that each portion oscillates in space and time, but are out of phase.

SOLUTION:
To compute the energy density of the wave, we can use the expression from the previous problem. However, in this case it is simpler to compute it using:

$$u(\mathbf{r}, t) = \frac{\epsilon_0}{2}\mathrm{Re}[\mathbf{E}_{14}]^2 + \frac{1}{2\mu_0}\mathrm{Re}[\mathbf{B}_{14}]^2 \tag{67}$$

$$= 2\epsilon_0 (E_1^0)^2 \cos^2(kz)\cos^2(\omega t) + 2\epsilon_0 (E_1^0)^2 \sin^2(kz)\sin^2(\omega t) \tag{68}$$

This shows that the two parts oscillate in space and time but are out of phase.